p-ADIC PROPERTIES FOR TRACES OF SINGULAR MODULI

PAUL JENKINS

ABSTRACT. We examine the p-adic properties of Zagier's traces $\mathrm{Tr}(d)$ of the singular moduli of discriminant $-d$. In a recent preprint, Edixhoven proved that if p is prime and $\left(\frac{-d}{p}\right) = 1$, then

$$\mathrm{Tr}(p^{2n}d) \equiv 0 \pmod{p^n}.$$

We compute an exact formula for $\mathrm{Tr}(p^{2n}d)$ which immediately gives Edixhoven's result as a corollary.

1. Introduction and statement of results

As usual, let $j(z)$ be the modular function for $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$j(z) = \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + 196884q + \cdots,$$

where $q = e^{2\pi i z}$.

Let $d \equiv 0, 3 \pmod 4$ be a positive integer, so that $-d$ is a negative discriminant. Denote by $\mathcal{Q}_d$ the set of positive definite integral binary quadratic forms $Q(x, y) = ax^2+bxy+cy^2 = [a, b, c]$ with discriminant $-d = b^2-4ac$, including imprimitive forms (if such exist). Write α_Q for the unique complex number in the upper half plane $\mathfrak{H}$ satisfying $Q(\alpha_Q, 1) = 0$. Values of j at the points α_Q are known as *singular moduli*. Because of the modularity of j, the singular modulus $j(\alpha_Q)$ depends only on the equivalence class of Q under the action of $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

Define $\omega_Q \in \{1, 2, 3\}$ as

$$(1.1) \qquad \omega_Q = \begin{cases} 2 & \text{if } Q \sim_\Gamma [a, 0, a], \\ 3 & \text{if } Q \sim_\Gamma [a, a, a], \\ 1 & \text{otherwise.} \end{cases}$$

Following Zagier [Z], we define the trace of the singular moduli of discriminant $-d$ as

$$(1.2) \qquad \mathrm{Tr}(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{j(\alpha_Q) - 744}{\omega_Q}.$$

Zagier related these traces of singular moduli to the coefficients of a certain weight 3/2 modular form. We let $M^!_{\lambda+1/2}$ be the space of weight $\lambda + 1/2$ weakly holomorphic

---

2000 *Mathematics Subject Classification.* 11F37.

The author thanks the National Science Foundation for the support of an NSF VIGRE fellowship.

modular forms on $\Gamma_0(4)$ with Fourier expansion

$$f(z) = \sum_{(-1)^\lambda n \equiv 0,1 \pmod 4} a(n)q^n.$$

Recall that a form is weakly holomorphic when its poles, if there are any, are supported on the cusps.

For any $0 < D \equiv 0, 1 \pmod 4$, let $g_D(z)$ be the unique element of $M^!_{3/2}$ with Fourier expansion

$$g_D(z) = q^{-D} + B_1(D, 0) + \sum_{0<d\equiv 0,3 \pmod 4} B_1(D, d)q^d. \tag{1.3}$$

For $0 \le d \equiv 0, 3 \pmod 4$, let $f_d(z)$ be the unique form in $M^!_{1/2}$ with expansion

$$f_d(z) = q^{-d} + \sum_{0<D\equiv 0,1 \pmod 4} A_1(D, d)q^D. \tag{1.4}$$

All of the coefficients $A_1(D, d)$ and $B_1(D, d)$ of the f_d and g_D are integers.

We apply Hecke operators (for definitions, see Section 3.1 of [O]) to the f_d and g_D and define

$$\begin{aligned} A_m(D, d) &= \text{the coefficient of } q^D \text{ in } f_d(z) \mid T_{\frac{1}{2}}(m^2), \\ B_m(D, d) &= \text{the coefficient of } q^d \text{ in } g_D(z) \mid T_{\frac{3}{2}}(m^2). \end{aligned} \tag{1.5}$$

Using this notation, Zagier ([Z], Theorem 5) proved that

$$A_m(D, d) = -B_m(D, d). \tag{1.6}$$

This duality between coefficients of modular forms of different weights is central to the result in this paper. Zagier also identified singular moduli with these coefficients by proving that if $-d < 0$ is a discriminant, then

$$\mathrm{Tr}(d) = -B_1(1, d). \tag{1.7}$$

Suppose that p is an odd prime and that s is a positive integer. When p is inert or ramified in particular quadratic number fields, Ahlgren and Ono [AO] proved many congruences for traces of singular moduli modulo p^s. In addition, they gave an elementary argument that $\mathrm{Tr}(p^2 d) \equiv 0 \pmod p$ when p splits in $\mathbb{Q}(\sqrt{-d})$. In a recent preprint, Edixhoven [E] extended their observation and proved that if $\left(\frac{-d}{p}\right) = 1$, then

$$\mathrm{Tr}(p^{2n} d) \equiv 0 \pmod{p^n}. \tag{1.8}$$

In another recent preprint, Boylan [B] exactly computes $\mathrm{Tr}(2^{2n} d)$, giving an analogous result when $p = 2$.

Remark. The aim of Edixhoven's paper is to show that p-adic geometry of modular curves can be used to study p-adic properties of traces of $f \in \mathbb{Z}[j]$, of which Zagier's trace is one. The congruences we cite are a special case of his more general result.

In this paper we obtain an exact formula for the coefficient $B_1(D, p^{2n}d)$ of $q^{p^{2n}d}$ in g_D, allowing us to obtain Edixhoven's congruences as a corollary.

Theorem 1.1. *If p is an odd prime and $-d, D \equiv 0, 1 \pmod 4$ and n are positive integers, then*

$$B_1(D, p^{2n}d) = p^n B_1(p^{2n}D, d) + \sum_{k=0}^{n-1} \left(\frac{D}{p}\right)^{n-k-1} \left(B_1\left(\frac{D}{p^2}, p^{2k}d\right) - p^{k+1} B_1\left(p^{2k}D, \frac{d}{p^2}\right)\right)$$
$$+ \sum_{k=0}^{n-1} \left(\frac{D}{p}\right)^{n-k-1} \left(\left(\left(\frac{D}{p}\right) - \left(\frac{-d}{p}\right)\right) p^k B_1(p^{2k}D, d)\right),$$

where $B_1(M, N) = 0$ if M or N is not an integer.

The following corollaries follow immediately.

Corollary 1.2. *For an odd prime p and positive integers n and $-d, D \equiv 0, 1 \pmod 4$, if $\left(\frac{D}{p}\right) = \left(\frac{-d}{p}\right) \neq 0$, then $B_1(D, p^{2n}d) = p^n B_1(p^{2n}D, d)$.*

Corollary 1.3. *If $\left(\frac{-d}{p}\right) = 1$, then $\mathrm{Tr}(p^{2n}d) = -p^n B_1(p^{2n}, d)$.*

Remark. These results can easily be extended to Zagier's generalized traces $\mathrm{Tr}_m(d)$.

2. Proof of Theorem

To give the proof of Theorem 1.1, we need the following formulas for the action of the Hecke operators, given in Section 6 of [Z]. For an odd prime p, we have

$$A_p(D, d) = pA_1(p^2D, d) + \left(\frac{D}{p}\right) A_1(D, d) + A_1\left(\frac{D}{p^2}, d\right), \tag{2.1}$$

$$B_p(D, d) = B_1(D, p^2d) + \left(\frac{-d}{p}\right) B_1(D, d) + pB_1\left(D, \frac{d}{p^2}\right). \tag{2.2}$$

Combining Zagier's formula (1.6) with equation (2.1), we get

$$B_p(D, d) = pB_1(p^2D, d) + \left(\frac{D}{p}\right) B_1(D, d) + B_1\left(\frac{D}{p^2}, d\right).$$

Apply equation (2.2) to get

$$B_1(D, p^2d) = pB_1(p^2D, d) + \left(\frac{D}{p}\right) B_1(D, d) + B_1\left(\frac{D}{p^2}, d\right) - \left(\frac{-d}{p}\right) B_1(D, d) - pB_1\left(D, \frac{d}{p^2}\right). \tag{2.3}$$

These observations alone suffice to prove Theorem 1.1.

Proof of Theorem 1.1. We prove the theorem by induction on n. The $n = 1$ case is just equation (2.3). For $n > 1$, assume the theorem holds up to $n - 1$.

Replacing d by $p^{2n-2}d$ in equation (2.3) gives

(2.4)
$$B_1(D,p^{2n}d) = p(B_1(p^2D,p^{2n-2}d) - B_1(D,p^{2n-4}d)) + B_1\left(\frac{D}{p^2},p^{2n-2}d\right) + \left(\frac{D}{p}\right)B_1(D,p^{2n-2}d).$$

Note that for $1 \le k \le n-2$, replacing D with $p^{2k}D$ and d with $p^{2n-2k-2}d$ in (2.3) gives

(2.5)
$$\begin{aligned} &p^k(B_1(p^{2k}D,p^{2n-2k}d) - B_1(p^{2k-2}D,p^{2n-2k-2}d)) \\ =&p^{k+1}(B_1(p^{2k+2}D,p^{2n-2k-2}d) - B_1(p^{2k}D,p^{2n-2k-4}d)). \end{aligned}$$

Making a similar replacement in (2.3) with $k = n-1$, we get

(2.6)
$$\begin{aligned} &p^{n-1}(B_1(p^{2n-2}D,p^2d) - B_1(p^{2n-4}D,d)) \\ =&p^nB_1(p^{2n}D,d) - p^nB_1\left(p^{2n-2}D,\frac{d}{p^2}\right) - p^{n-1}\left(\frac{-d}{p}\right)B_1(p^{2n-2}D,d). \end{aligned}$$

Using (2.5) $n-2$ times and (2.6), we obtain

(2.7)
$$\begin{aligned} &p(B_1(p^2D,p^{2n-2}d) - B_1(D,p^{2n-4}d)) \\ =&p^nB_1(p^{2n}D,d) - p^nB_1\left(p^{2n-2}D,\frac{d}{p^2}\right) - p^{n-1}\left(\frac{-d}{p}\right)B_1(p^{2n-2}D,d). \end{aligned}$$

Substituting (2.7) in (2.4) gives

$$\begin{aligned} B_1(D,p^{2n}d) =&p^nB_1(p^{2n}D,d) - p^nB_1\left(p^{2n-2}D,\frac{d}{p^2}\right) - p^{n-1}\left(\frac{-d}{p}\right)B_1(p^{2n-2}D,d) \\ &+ B_1\left(\frac{D}{p^2},p^{2n-2}d\right) + \left(\frac{D}{p}\right)B_1(D,p^{2n-2}d). \end{aligned}$$

Apply the induction hypothesis to expand the last term; after simplifying, the theorem follows. □

References

[AO] S. Ahlgren and K. Ono, *Arithmetic of singular moduli and class equations*, Compositio Mathematica **141** (2005), pages 293-312.

[B] M. Boylan, *2-adic properties of Hecke traces of singular moduli.* Preprint, 2004.

[E] B. Edixhoven, *On the p-adic geometry of traces of singular moduli.* Preprint, 2005, arXiv:math.NT/0502212.

[O] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series **102**, Amer. Math. Soc., Providence, R. I., 2004.

[Z] D. Zagier, Traces of singular moduli, *Motives, Polylogarithms and Hodge Theory*, Part I. International Press Lecture Series (Eds. F. Bogomolov and L. Katzarkov), International Press, 2002, pages 211-244.

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

E-mail address: pjenkins@math.wisc.edu