RESEARCH STATEMENT

PAUL M. JENKINS

Introduction

My research has focused on modular forms and Maass forms and their applications in number theory. A modular form is a holomorphic function on the complex upper half plane satisfying a transformation law

$$f\left(\frac{az+b}{cz+d}\right) = \varepsilon(a,b,c,d)(cz+d)^k f(z),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Here $\varepsilon(a,b,c,d)$ has absolute value 1, the weight k is in $\frac{1}{2}\mathbb{Z}$, and f satisfies certain growth conditions. Writing $q = e^{2\pi i z}$, a modular form $f(z)$ has a Fourier expansion

$$f(z) = \sum_{n=n_0}^{\infty} a(n)q^n.$$

Maass forms are similar to modular forms, but are not holomorphic, and are required to be eigenfunctions of the hyperbolic Laplacian.

Modular forms are objects of intense interest in number theory; most famously, they played a vital role in Wiles' proof of Fermat's Last Theorem. The coefficients $a(n)$ of these forms play many different roles, appearing, for example, in the study of points on elliptic curves over finite fields, representations of integers by quadratic forms, values of the partition function, class numbers, the Birch and Swinnerton-Dyer conjecture, and values of L-functions.

These coefficients make a prominent and interesting appearance in work of Borcherds. To explain, recall that many modular forms have simple infinite product expansions. For example, the weight 12 modular form $\Delta(z)$ has the expansion

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

The weight 4 Eisenstein series $E_4(z)$ has the more complicated expansion

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n = (1-q)^{-240}(1-q^2)^{26760}(1-q^3)^{-4096240}\cdots.$$

In his 1994 ICM lecture, Borcherds [2],[3] gave a striking description of the exponents in the infinite product expansion of modular forms with Heegner divisor, of which these are two examples, showing that the exponents in such an expansion appear as coefficients in the Fourier expansion of another modular form of weight 1/2. In particular, the product expansions of Δ and E_4 above correspond to the weight 1/2 modular forms

$$12\theta = 12 + 24q + 24q^4 + 24q^9 + \cdots,$$

$$G(z) = q^{-3} + 4 - 240q + 26760q^4 - 85995q^5 + \cdots - 4096240q^9 + \cdots.$$

The exponents of the $(1-q^n)$ in the infinite product expansions of Δ and E_4 equal the coefficients of q^{n^2} in 12θ and $G(z)$. Borcherds also proved that this map between spaces of modular forms is an isomorphism.

Theorem 1 (Borcherds)**.** *Let $f(z)$ be an integer weight meromorphic modular form on* $\mathrm{SL}_2(\mathbb{Z})$ *with Heegner divisor, integer coefficients, and leading coefficient 1. Then*

$$f(z) = q^{-h} \prod_{n=1}^{\infty} (1-q^n)^{A(n^2)},$$

where $A(D)$ is the coefficient of q^D in a certain nearly holomorphic modular form of weight $1/2$ *for the group* $\Gamma_0(4)$*.*

Borcherds' results are in fact more general, using automorphic forms on orthogonal groups. One question posed in his paper is whether a proof of this theorem exists that uses only the classical theory of modular forms. Zagier [28] answered this in the affirmative, and his work has inspired my research to date. Zagier's results are phrased in terms of singular moduli.

The usual j-function is the modular function for $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion

$$j(z) = q^{-1} + 744 + 196884q + \cdots .$$

Values of j at quadratic irrational points of discriminant $-d < 0$ are known as *singular moduli*. Singular moduli are algebraic integers which have been extensively studied in number theory; for example, they generate ring class field extensions of imaginary quadratic number fields.

Zagier [28] initiated the study of $\mathrm{Tr}(d)$, the function giving the algebraic trace of a singular modulus of discriminant $-d$, and its generalization $\mathrm{Tr}_m(d)$, and showed that these traces appear as coefficients of a certain weight $3/2$ nearly holomorphic modular form.

To make this more precise, define g_D as the unique weight $3/2$ modular form on $\Gamma_0(4)$ with Fourier expansion

$$g_D(z) = q^{-D} + B_1(D,0) + \sum_{0<d\equiv 0,3 \pmod 4} B_1(D,d)q^d.$$

Applying the weight $3/2$ Hecke operator $T_{\frac{3}{2}}(m^2)$, let

$$B_m(D,d) \;=\; \text{the coefficient of } q^d \text{ in } g_D(z) \;\big|\; T_{\frac{3}{2}}(m^2).$$

Zagier proved the following theorem relating traces of singular moduli and the coefficients of the g_D.

Theorem 2 (Zagier)**.** *If $m \geq 1$ and $-d < 0$ is a discriminant, then*

$$\mathrm{Tr}_m(d) = -B_m(1,d).$$

By exhibiting a striking duality between these coefficients $B_m(D,d)$ and the coefficients of another sequence of modular forms, this time of weight $1/2$, Zagier was able to give a new proof of Borcherds' infinite product formulas (Theorem 1) with this theorem. However, although Theorem 2 implies Borcherds' theorem and provides the desired "modular form" proof, it does not provide much new information about the $\mathrm{Tr}_m(d)$. For example, it does not give exact formulas for calculating the coefficients $B_m(D,d)$, or tell us about their p-divisibility.

My results

Combining properties of Kloosterman sums with the duality between coefficients of modular forms of different weights, and using an expression for $\mathrm{Tr}_m(d)$ due to Duke [11], I gave a new proof [16] of Theorem 2, thus giving a new "modular form" proof of Borcherds' theorem.

An essential element in this new proof is an explicit expression for the coefficients $B_1(D,d)$ of the modular form g_D. This formula was derived in my work with Bruinier and Ono [8] from certain results about weakly holomorphic Maass-Poincaré series. More precisely, if $H(d)$

is the Hurwitz class number for the discriminant $-d$, $\sigma(n)$ is the sum of the divisors of n, and $K(m,n;c)$ is a generalized Kloosterman sum, this expression for the coefficients, together with Hecke operator relations, give the following exact formula for the traces $\mathrm{Tr}_m(d)$.

Theorem 3 (Bruinier-J.-Ono [8], Theorem 1.2)**.** *For discriminants $-d<0$, we have*

$$\mathrm{Tr}_m(d) = -24\sigma(m)H(d) + (1+i)\sum_{n|m}\sum_{\substack{c>0\\ c\equiv 0\ (4)}} (1+\delta_{\mathrm{odd}}(c/4))\frac{K(-n^2,d;c)}{\sqrt{c}}\sinh\left(\frac{4\pi n}{c}\sqrt{d}\right).$$

For a fundamental discriminant $-d$, the Hilbert class polynomial $\mathcal{H}_d(x)$ is the polynomial whose roots are the singular moduli of discriminant $-d$. These irreducible polynomials generate the Hilbert class field of $\mathbb{Q}(\sqrt{-d})$, and the problem of computing them and their roots has a long history (for example, see [10] or [4]). Since the coefficients of these polynomials are symmetric functions of singular moduli, the Newton-Girard formulas can be used to write the coefficients in terms of sums of powers of singular moduli, which are easily computed from the $\mathrm{Tr}_m(d)$ for $1\le m\le H(d)$. Thus, Theorem 3 has the following corollary.

Corollary 4. *If $-d<0$ is a fundamental discriminant, then we have an exact formula for the Hilbert class polynomial $\mathcal{H}_d(x)$.*

Zagier also defined "twisted" traces of singular moduli by multiplying values of j by a genus character χ_D before adding them together. If D is a fundamental discriminant and α is a quadratic irrational of discriminant $-Dd$, this twisted trace is related to the trace of the algebraic integer $j(\alpha)$ from the Hilbert class field of $\mathbb{Q}(\sqrt{-Dd})$ to its real quadratic subfield $\mathbb{Q}(\sqrt{D})$. Values of these twisted traces are also encoded in the coefficients of the modular forms g_D, in a manner similar to Theorem 2. Bringmann and Ono [6] have extended this description of twisted traces to more general weights by using Maass-Poincaré series.

With the following theorem, I generalized a result of Duke [11] on standard traces to get a new exact formula [17] for twisted traces.

Theorem 5 (J. [17], Theorem 1.5)**.** *If $D,-d\equiv 0,1 \pmod 4$ are a positive fundamental discriminant and a negative discriminant, respectively, with $D>1$, and $m\ge 1$ is an integer, we have*

$$\mathrm{Tr}_m(D,d) = \sum_{\substack{c\equiv 0(4)\\ c>0}}\sum_{\substack{x(c)\\ x^2\equiv -Dd(c)}} \chi_D\left(\frac{c}{4},x,\frac{x^2+Dd}{c}\right) e\left(\frac{2mx}{c}\right)\sinh\left(\frac{4\pi m\sqrt{dD}}{c}\right).$$

Suppose that p is an odd prime and that s is a positive integer. When p is inert or ramified in particular quadratic number fields, Ahlgren and Ono [1] proved many congruences for traces of singular moduli modulo p^s. In addition, they gave an elementary argument that $\mathrm{Tr}(p^2d)\equiv 0 \pmod p$ when p splits in $\mathbb{Q}(\sqrt{-d})$ (that is, when $\left(\frac{-d}{p}\right)=1$). In a recent preprint, Edixhoven [13] extended their observation and proved that if $\left(\frac{-d}{p}\right)=1$, then

$$\mathrm{Tr}(p^{2n}d)\equiv 0 \pmod{p^n}.$$

In another paper, Boylan [5] exactly computed $\mathrm{Tr}(2^{2n}d)$, giving a stronger result when $p=2$.

Using the duality between modular forms of weights $3/2$ and $1/2$, I proved the following identity among the coefficients $B_m(D,d)$ of the modular forms g_D.

Theorem 6 (J. [16], Theorem 1.1)**.** *If p is an odd prime and $-d, D \equiv 0, 1 \pmod 4$ and n are positive integers, then*

$$B_1(D, p^{2n}d) = p^n B_1(p^{2n}D, d) + \sum_{k=0}^{n-1} \left(\frac{D}{p}\right)^{n-k-1} \left(B_1\left(\frac{D}{p^2}, p^{2k}d\right) - p^{k+1} B_1\left(p^{2k}D, \frac{d}{p^2}\right)\right)$$
$$+ \sum_{k=0}^{n-1} \left(\frac{D}{p}\right)^{n-k-1} \left(\left(\left(\frac{D}{p}\right) - \left(\frac{-d}{p}\right)\right) p^k B_1(p^{2k}D, d)\right),$$

where $B_1(M, N) = 0$ if M or N is not an integer.

Taking $D = 1$ and recalling that $\mathrm{Tr}(d) = -B_1(1, d)$, we have the following precise formula which implies Edixhoven's congruences.

Corollary 7. *If $\left(\frac{-d}{p}\right) = 1$, then* $\mathrm{Tr}(p^{2n}d) = -p^n B_1(p^{2n}, d) \equiv 0 \pmod{p^n}$.

Future research

Singular moduli may be generalized by replacing the j-function with a modular function of higher level, such as the Hauptmodul associated to other groups of genus zero, as in work of Kim [18] [19]. Osburn [26] proved that Ahlgren and Ono's congruence extends to this case, suggesting that an analogue of Theorem 6 will also hold for higher levels and give information about the p-divisibility of traces of $p^{2n}d$ for these functions. I expect to work out such a generalization in the near future.

In addition, the entire framework underlying these results may be generalized by examining modular forms of weights other than 3/2 or 1/2. Bringmann and Ono's work [6] using Maass-Poincaré series for these half-integral weight modular forms shows that similar duality theorems hold, and Zagier discussed the interpretation of the coefficients of these forms as traces of non-holomorphic modular functions. For example, the coefficients of certain modular forms of weights 5/2 and −1/2 correspond to traces of the function

$$K(z) = \frac{5}{6} j(z) - 576 + \frac{E_2^*(z)E_4(z)E_6(z)}{6\Delta(z)},$$

where $E_2^*(z)$ is the non-holomorphic Eisenstein series of weight 2. A generalization of Theorem 6 would provide information about the p-adic properties of these traces, and should provide an arithmetic interpretation in terms of isomorphism classes of elliptic curves with complex multiplication. My preliminary work in this direction seems to confirm that an analogue of Theorem 6 is true in much greater generality, and I hope to clarify this speculation. Such a generalization would involve values of exceptional harmonic weak Maass forms at CM points.

Recently, Bruinier and Funke [7] obtained results on traces of modular functions that hold in much greater generality; specifically, they removed the restriction to modular curves of genus zero inherent in Zagier's work. Their method involves integrating arbitrary modular functions of weight zero against certain theta series to obtain modular forms whose positive Fourier coefficients are the desired traces. In addition, they obtain geometric interpretations for the constant term and negative Fourier coefficients, and recover Zagier's results as special cases. I am interested in using the techniques and ideas I have developed in my research thus far to study the p-adic properties of these traces and their arithmetic interpretations.

Although my work to date has been mostly related to Borcherds' and Zagier's work and to singular moduli, in the near future I plan to explore other questions. I am interested in the relationship between central critical values of quadratic twists of even weight modular L-functions and the coefficients of associated cusp forms. This relates to the Birch and Swinnerton-Dyer conjecture, one of the Clay Mathematics Institute Millennial Prize problems. The conjecture is as follows.

Conjecture 8 (Birch and Swinnerton-Dyer). *For an elliptic curve $E/\mathbb{Q}$,*

$$\operatorname{ord}_{s=1} L(E, s) = \operatorname{rank}(E).$$

Moreover, if E_{tor} is the torsion subgroup of E over $\mathbb{Q}$, $Ш(E)$ is the Shafarevich-Tate group of E, $\operatorname{Tam}(E)$ is the Tamagawa number for E, $\Omega(E)$ is the real period of E, and $R(E)$ is the regulator of $E/\mathbb{Q}$, then

$$\lim_{s\to 1}\left((s-1)^{-\operatorname{rank}(E)}L(E,s)\right) = \frac{2^{\operatorname{rank}(E)}\cdot |Ш(E)|\cdot \operatorname{Tam}(E)\cdot \Omega(E)\cdot R(E)}{|E_{\text{tor}}^2|}.$$

Although this problem remains open, there have been some significant results in its direction. The most famous result is due to Kolyvagin [23], who proved that if the order of the L-function at $s = 1$ is 0 or 1, then

$$\operatorname{ord}_{s=1}(L(E, s)) = \operatorname{rank}(E) \textit{ and } |Ш(E)| < \infty.$$

This result was at first conditional, but the restrictions were removed by work of Gross and Zagier [14] and of Bump, Friedberg and Hoffstein [9] on the nonvanishing of central critical values of quadratic twists of L-functions. Further work in this direction has been done by Murty and Murty [24], Iwaniec [15], Waldspurger [27], and Ono and Skinner [25].

Gross and Zagier's work relates the central value of an L-function to the canonical height of a Galois trace of a Heegner point; a natural question, then, would be whether the central critical values themselves can be interpreted as traces. Work of Kohnen and Zagier [21], [20], [22] and Waldspurger shows that central critical values of quadratic twists of L-functions can be written in terms of the squares of certain Fourier coefficients of half integral weight cusp forms. Using recent work of Bringmann and Ono [6] as a prototype, it should be possible to adapt theorems on Maass-Poincaré series to cuspidal Poincaré series to describe these L-values in terms of traces of some sort of modular invariant evaluated at CM points or their equivalent. I intend to explore this relationship thoroughly.

I am also interested in learning more about equidistribution results, such as those arising in Duke's work [12]. His theorems have interesting applications to the asymptotic behavior of $\operatorname{Tr}(d)$. More specifically, we may approximate the j-function by the q^{-1} term in its Fourier expansion and examine the average difference $\frac{G(d)-\operatorname{Tr}(d)}{H(d)}$ between traces of singular moduli and their approximations $G(d)$ arising from this truncation. Some care is required to define $G(d)$ properly, as some "small" CM points are actually counted with multiplicity two. Since $H(163) = 1$, computing this difference for $d = 163$ gives the famous classical observation that

$$e^{\pi\sqrt{163}} - 262537412640768744 = -0.0000000000007499\ldots$$

and suggests that this average should be "small". Duke was able to use methods he developed in [12] concerning the distribution of CM points to prove the following theorem [11], where $\zeta(s)$ is the Riemann zeta function.

Theorem 9 (Duke). *As $-d$ ranges over negative fundamental discriminants, we have*

$$\lim_{-d\to-\infty}\frac{\operatorname{Tr}(d)-G(d)}{H(d)} = -24 = \frac{2}{\zeta(-1)}.$$

I plan to develop several results of this kind with respect to generic CM points and singular moduli problems; the idea will be to determine when such limiting distributions come from values of zeta functions.

References

1. S. Ahlgren and K. Ono, *Arithmetic of singular moduli and class polynomials*, Compositio Math. **141** (2005), 293–312.
2. R. E. Borcherds, *Automorphic forms on* $\mathrm{O}_{s+2,2}(\mathbf{R})$ *and infinite products*, Invent. Math. **120** (1995), no. 1, 161–213.
3. ______, *Automorphic forms on* $\mathrm{O}_{s+2,2}(\mathbf{R})^+$ *and generalized Kac-Moody algebras*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 744–752.
4. A. Borel, S. Chowla, C. S. Herz, K. Iwasawa, and J.-P. Serre, *Seminar on complex multiplication*, Seminar held at the Institute for Advanced Study, Princeton, N.J., 1957-58. Lecture Notes in Mathematics, No. 21, Springer-Verlag, Berlin, 1966.
5. M. Boylan, *2-adic properties of Hecke traces of singular moduli*, Math. Res. Letters, accepted for publication.
6. K. Bringmann and K. Ono, *Arithmetic properties of coefficients of half-integral weight Maass-Poincaré series*, preprint.
7. J. Bruinier and J. Funke, *Traces of CM-values of modular functions*, J. Reine Angew. Math., accepted for publication.
8. J. Bruinier, P. Jenkins, and K. Ono, *Hilbert class polynomials and traces of singular moduli*, Mathematische Annalen, accepted for publication.
9. D. Bump, S. Friedberg, and J. Hoffstein, *Nonvanishing theorems for* l*-functions of modular forms and their derivatives*, Invent. Math. **102** (1990), 543–618.
10. D. A. Cox, *Primes of the form* $x^2 + ny^2$, A Wiley-Interscience Publication, John Wiley & Sons Inc., New York, 1989.
11. W. Duke, *Modular functions and the uniform distribution of CM points*, Mathematische Annalen, accepted for publication.
12. ______, *Hyperbolic distribution problems and half-integral weight Maass forms*, Invent. Math. **92** (1988), 73–90.
13. B. Edixhoven, *On the* p*-adic geometry of traces of singular moduli*, preprint, 2005, arXiv:math.NT/0502212.
14. B. H. Gross and D. B. Zagier, *Heegner points and derivatives of* L*-series*, Invent. Math. **84** (1986), no. 2, 225–320.
15. H. Iwaniec, *On the order of vanishing of modular* L*-functions at the critical point*, Sém. Théor. Nombres Bordeaux (2) **2** (1990), no. 2, 365–376.
16. P. Jenkins, *Kloosterman sums and traces of singular moduli*, J. Number Th., accepted for publication.
17. ______, p*-adic properties for traces of singular moduli*, International Journal of Number Theory **1** (2005), 103–107.
18. C. H. Kim, *Traces of singular values and Borcherds products*, preprint.
19. ______, *Borcherds products associated to certain Thompson series*, Compositio Math. **140** (2004), 541–551.
20. W. Kohnen, *Newforms of half-integral weight*, J. Reine Angew. Math. **333** (1982), 32–72.
21. ______, *Fourier coefficients of modular forms of half integral weight*, Math. Ann. **271** (1985), 237–268.
22. W. Kohnen and D. Zagier, *Values of* L*-series of modular forms at the center of the critical strip*, Invent. Math. **64** (1981), no. 2, 175–198.
23. V. A. Kolyvagin, *Finiteness of* $E(\mathbf{Q})$ *and the tate-shafarevich group of* $(e, \mathbf{q})$ *for a subclass of Weil curves*, Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), no. 3, 522–540, 670–671.
24. M. R. Murty and V. K. Murty, *Mean values of derivatives of modular* L*-series*, Ann. of Math. (2) **133** (1991), no. 3, 447–475.
25. K. Ono and C. Skinner, *Non-vanishing of quadratic twists of modular* L*-functions*, Invent. Math. **134** (1998), no. 3, 651–660.
26. R. Osburn, *Congruences for traces of singular moduli*, Ramanujan J., accepted for publication.
27. J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*, J. Math. Pures Appl. (9) **60** (1981), no. 4, 375–484.
28. D. Zagier, *Traces of singular moduli*, Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), Int. Press Lect. Ser., vol. 3, Int. Press, Somerville, MA, 2002, pp. 211–244.