# Math 313 - Linear Algebra
# Proof writing

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In order to disprove the assertion that all crows are black,
one white crow is sufficient.

- William James

# Theorems and Proofs

A **theorem** is a mathematical statement that provides a conclusion, provided that a set of specific assumptions holds.

### Theorem

If certain assumptions hold, then a specific conclusion will also hold.

The proof of a theorem consists of statements, each of which is

- an assumption,
- a conclusion, which clearly follows from an assumption or a previously proved result.

# (Dis)proving a mathematical statement

To disprove a mathematical statement, it is often enough to provide a single counterexample showing that the statement is false.

**Example:** To disprove the statement

*All prime numbers are odd.*

it suffices to find a single prime number which is even (e.g. 2).

Showing that a mathematical statement is true requires a proof.

**Example:** The statement

*Any even number* $> 2$ *can be written as the sum of two primes.*

would require a proof (if true), or a counterexample (if false).

# Types of mathematical statements

Single implication:

If $P$ holds, then $Q$ holds (written $P \implies Q$).

Note: $P \implies Q$ is *not* the same as $Q \implies P$ (one may be true, while the other may be false).

**Example:** If $n \in \mathbb{N}$ is divisible by 6, then $n$ is divisible by 3.

on the other hand

If $n$ is divisible by 3, then $n$ may or may not be divisible by 6.

# Proof types - direct proof

Start with the assumptions, and try to reach the desired conclusion directly.

**Examples:**

Theorem

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a set of $p$ vectors in $\mathbb{R}^n$, and $n < p$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

Theorem

If a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ contains the zero vector $\mathbf{0}$, then is is linearly dependent.

# Proof types - proof by contrapositive

The statement ($P \implies Q$) is equivalent to (not $Q \implies$ not $P$).

**Examples:** If $n \in \mathbb{N}$ is divisible by 6, then $n$ is divisible by 3.

is the same as

If $n$ is not divisible by 3, then $n$ is not divisible by 6.

Proof by contrapositive: Prove (not $Q \implies$ not $P$) instead of ($P \implies Q$).

Start by assuming (not $Q$), and try to prove (not $P$).

**Theorem**

If the product $r \cdot s \geq 100$, then either $r \geq 10$ or $s \geq 10$.

# Proof types - proof by contradiction

Proof by contradicton: Start by assuming

$$\text{both } P \text{ and (not } Q\text{)}$$

and try to to derive a contradiction.

Since having both $P$ and (not $Q$) leads to a contradiction, whenever $P$ is true $Q$ must also be true.

**Theorem**
The sum of a rational number and an irrational number is irrational.

# Types of mathematical statements

## If and only if:

$P$ if and only if $Q$ (written $P \iff Q$) is the same as

$$\textit{both } (P \implies Q) \textit{ and } (Q \implies P).$$

To prove $P \iff Q$, we must prove both $P \implies Q$ and $Q \implies P$.

**Theorem**

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent if and only if some $\mathbf{v}_j$ can be written as a linear combination of the other vectors.

# Proof types - proof by induction

Suppose we want to prove that something is true for all natural numbers $n \geq 1$.

First step: Prove that it is true for $n = 1$.

Inductive step: Assume that the statement is true for some $n \geq 1$. Prove that it must also be true for $n + 1$.

Then the statement must be true for all $n \geq 1$.

## Theorem

For any $n \in \mathbb{N}$, we have $1 + 2 + \cdots + (n-1) + n = \dfrac{n(n+1)}{2}$.