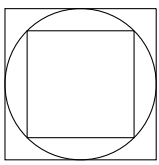
Chapter IX

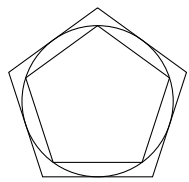
Introduction to Analysis

The only way to discover the limits of the possible is to go beyond them into the impossible. Arthur C. Clarke

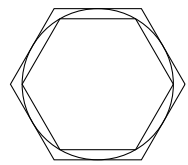
In the third century BC, the Greek mathematician Archimedes used the "method of exhaustion" to estimate the circumference of a circle of diameter 1, and thus estimate the value of π. His method involved inscribing a regular n-gon inside the circle, circumscribing the circle by a regular n-gon, and bounding the circumference of the circle between the perimeters of the two n-gons. For example, taking $n = 4$, $n = 5$, and $n = 6$, we get the following approximations.



$2.8284 < \pi < 4.0000$



$2.9389 < \pi < 3.6327$



$3.0000 < \pi < 3.464$

As n gets larger the approximations get better; when $n = 100$ we get

$$3.141 < \pi < 3.143,$$

and when $n = 1000$ we get

$$3.141587 < \pi < 3.141603.$$

This computation was among the first uses in antiquity of the idea of a limit; however, it would be nearly two millennia before the concept of limit was formally defined and given a logical foundation.

Newton and Leibniz used a concept of limit in the development of calculus, but it was not until around 1820 that Bolzano and Cauchy formalized the definition of limit. It was even later when it was finally written in the way most mathematicians now use limits.

34 Sequences

The infinite list of numbers

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$$

is an example of what we shall call a sequence. Sequences arise naturally in many contexts. For instance, you might measure the speed of a race car every second and produce a list of speeds. Or you could measure, over time, the temperature of heated metal. Maybe your list of numbers is the total population in a bacterial culture, measured every morning in the lab.

In all these cases, the numbers give us a brief glimpse at a process that could continue on forever. Extrapolating from the data, we might make a guess about how the sequence behaves, or perhaps fit it to a nice function. For instance, you might guess that the list of numbers above comes from the function $1/n$, as n ranges over the natural numbers (and you'd be right!).

A fundamentally important question we can ask is: Where are these numbers headed? Scientists use the mathematical theory developed in this section to determine the eventual behavior, or *limit*, of such sequences.

34.A What are sequences, exactly?

The formal definition of a sequence is as follows.

Definition 34.1. A *sequence* is a function $f\colon \mathbb{N} \to \mathbb{R}$. The image $f(n)$ of $n \in \mathbb{N}$ is called the nth *term* of the sequence. Often we will write the terms of a sequence with subscripts; i.e., we write them as

$$a_1, a_2, a_3, \dots$$

where $a_n = f(n)$.

Example 34.2. It is important to be able to pass back and forth between a list of numbers and the function that defines the list.

For example, define a function $f\colon \mathbb{N} \to \mathbb{R}$ by the rule $f(n) = 2n$. Thus, the nth term of our sequence is $a_n = 2n$, and the first few terms are given as follows: $a_1 = 2,\ a_2 = 4,\ a_3 = 6,\ a_4 = 8,\ a_5 = 10, \dots$.

On the other hand, if you are given the list of numbers $2, 4, 8, 16, \dots$, then you might guess that this sequence arises from the function $g\colon \mathbb{N} \to \mathbb{R}$ given by the rule $g(n) = 2^n$. △

Example 34.3. Try to figure out the rule for the following sequences:

(1) $-1, 1, -1, 1, -1, 1, \dots$,
(2) $1, 3, 5, 7, \dots$,
(3) $1, 0, 0, 0, 0, \dots$, and
(4) $-10, 18397863, 2, 939, -10383, \dots$.

Here are some possible answers:
(1) $a_n = (-1)^n$,
(2) $a_n = 2n - 1$,
(3) $a_n = 0^{n-1}$ (or you could give a piecewise definition), and
(4) not enough information to find a pattern! △

Definition 34.4. If we wish to refer to a sequence $a_1, a_2, a_3, \ldots$, we may use the notation

$$(a_n)_{n\in\mathbb{N}}.$$

Example 34.5. Define $a_n = 2^n$. If we wish to refer to the sequence

$$a_1, a_2, a_3, \ldots$$

we could write $(a_n)_{n\in\mathbb{N}}$ or $(2^n)_{n\in\mathbb{N}}$. Both notations would refer to the sequence

$$2, 4, 8, \ldots. \quad \triangle$$

34.B Arithmetic sequences

Each of the following sequences have something in common. See if you can discover what it is.

$$1, 3, 5, 7, 9, \ldots$$
$$13, 8, 3, -2, -7, \ldots$$
$$6, 9, 12, 15, 18, \ldots$$

Answer: Each term in the sequence is a fixed distance from the previous term. In the first sequence, the terms jump by adding 2, in the second sequence they jump by adding -5, and in the last sequence the terms jump by adding 3.

These types of sequences are so common we give them a special name.

Definition 34.6. Let $c, d \in \mathbb{R}$. The sequence given by the formula

$$a_n = c + (n-1)d$$

for each $n \in \mathbb{N}$ is called the *arithmetic sequence* with first term c and common difference d.

Example 34.7. The arithmetic sequence with first term 2 and common difference 2 has terms

$$a_1 = 2,\ a_2 = 4,\ a_3 = 6,\ a_4 = 8,\ a_5 = 10, \ldots$$

The nth term is given by the formula $a_n = 2 + (n-1)2 = 2n$. △

Can you find the first six terms of the arithmetic sequence with first term π and common difference $-e$?

34.C Geometric sequences

There is a second very common type of sequence. Look to see if you can find what the following sequences have in common.

$$1, 2, 4, 8, 16, 32, \dots$$
$$1, 1/2, 1/4, 1/8, 1/16, \dots$$
$$3, -6, 12, -24, 48, \dots$$

Answer: Each term in the sequence is a fixed multiple of the previous term. In the first sequence we multiply each term by 2 to get the next term, in the second sequence we multiply by $1/2$, and in the third sequence we multiply by -2.

Definition 34.8. Let $c, r \in \mathbb{R}$. The sequence given by the formula

$$a_n = c \cdot r^{n-1}$$

for each $n \in \mathbb{N}$ is called the *geometric sequence* with first term c and common ratio r.

Example 34.9. The geometric sequence with first term 4 and common ratio $1/10$ is given by the formula

$$a_n = 4\left(\frac{1}{10}\right)^{n-1}$$

and the first few terms are

$$4, 4/10, 4/100, \dots$$ △

34.D Sequences and their limits

In this subsection we will discuss what it means for a sequence to approach a limit. To give an example, consider the sequence defined by the rule $a_n = \frac{1}{n}$, given at the beginning of this section. This sequence begins

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$$

It appears that the limit should be 0; the terms are getting closer and closer to 0.

We will now give the formal definition of what it means for a sequence to approach a limit. This definition is quite complicated, so we will explain the true *meaning* behind the symbols afterwards.

Definition 34.10. Given a sequence $(a_n)_{n\in\mathbb{N}}$, we say that it *converges to* $L \in \mathbb{R}$ if the following holds:

$$\forall \varepsilon \in \mathbb{R}_{>0},\ \exists N \in \mathbb{R},\ \forall n \in \mathbb{N},\ n > N \Rightarrow |a_n - L| < \varepsilon. \tag{34.11}$$

In this case, we say that L is a *limit* of the sequence.

If a sequence $(a_n)_{n\in\mathbb{N}}$ converges to some $L \in \mathbb{R}$, we say that the sequence *converges* or that it is a *convergent sequence*. We write $\lim_{n\to\infty} a_n = L$.

If a sequence does not converge to any limit, we say that the sequence *diverges*, or that it is a *divergent sequence*.

We will spend the rest of this section studying this definition and coming to an understanding of what it means. We begin by peeling off each of the quantifiers.

What is ε? The first quantifier is "$\forall \varepsilon \in \mathbb{R}_{>0}$." The variable ε is used to help us measure how close the sequence gets to the limit L. We want to be able to prove that our sequence can get *arbitrarily* close to the limit L. Thus, we want to prove that our sequence eventually gets within a distance of $1/100$ of the limit, but also eventually within a distance of $1/1000$ of the limit, and eventually within $1/10000$, and so forth. Thus, we do not just take $\varepsilon = 1/100$, we allow it to be *any* positive constant.

What is N? The second quantifier is "$\exists N \in \mathbb{R}$." The variable N helps us tell how far along the sequence we must go, so that after that point the sequence stays within a distance of ε from the limit.

For instance, again consider the sequence $1, 1/2, 1/3, \ldots$. When $\varepsilon = 1/2$, how far along the sequence do we need to travel until it stays within a distance $1/2$ of the limit $L = 0$? We see that by the time we reach the second term, all of the rest of the terms are within a distance of $1/2$ from 0. When $\varepsilon = 1/100$, we now must take $N = 100$. For even smaller values of ε, we have to take larger values of N so that the sequence stays that close to the limit.

Note that the second quantifier is *existential*. This means that you must *fix* a specific value of N (depending on ε) that will satisfy the definition of limit. This value of N will usually be found in scratch work, outside the proof. This is often the hardest part of a limit proof, and we will show how this is to be done shortly.

What is n? The third and final quantifier is "$\forall n \in \mathbb{N}$." The variable n is just one of the subscripts in our sequence.

What does the premise of the implication, $n > N$, say? The condition $n > N$ just tells us that we will only look at the terms in the sequence past N. When looking at limits, we really only care about what *eventually* happens.

What does the conclusion of the implication, $|a_n - L| < \varepsilon$, say? The condition $|a_n - L| < \varepsilon$ is just an easy way of saying that the nth term of our sequence is within a distance of ε from the limit L. Equivalently, by removing the absolute value signs, we may write $L - \varepsilon < a_n < L + \varepsilon$.

Proving Limits. Every proof of a limit for a sequence will look essentially the same. First you must deal with each of the quantified variables. The universal (for all) variables must be left arbitrary. The existential variables must be fixed, but *only*

in terms of previous variables. The implication is usually dealt with directly.

Proof outline.
Let $\varepsilon > 0$.
Fix $N = \boxed{\text{found from scratch work}} \in \mathbb{R}$.
Let $n \in \mathbb{N}$.
Assume $n > N$.
Do some work (usually by reversing the scratch work).
Conclude $|a_n - L| < \varepsilon$. □

We will show how this is accomplished, by proving the following result.

Theorem 34.12.
$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Scratch. This work should *not* appear on your homework, or in your proof.

Start with the conclusion $|a_n - L| < \varepsilon$. We know that $a_n = 1/n$ and $L = 0$. So the inequality becomes $|1/n - 0| < \varepsilon$. In other words $|1/n| < \varepsilon$. Since $1/n$ is positive, the absolute value signs disappear, and we have $1/n < \varepsilon$, or in other words $n > 1/\varepsilon$. This will be our value for N. ★

We are now ready to do the proof.

Proof. Let $\varepsilon > 0$. Fix $N = 1/\varepsilon \in \mathbb{R}$. Let $n \in \mathbb{N}$. Assume $n > N$. Thus, $n > 1/\varepsilon$. Taking reciprocals (noting that both sides of the inequality are positive), we get $1/n < \varepsilon$. Hence,
$$|a_n - L| = |1/n - 0| = 1/n < \varepsilon$$
as desired. □

Let's do another example.

Proposition 34.13. *Given the sequence* $(a_n)_{n\in\mathbb{N}}$ *defined by* $a_n = 1 - \frac{3}{n}$, *then*
$$\lim_{n\to\infty} a_n = 1.$$

Scratch. Start with $|a_n - L| < \varepsilon$. Plugging in the values for a_n and L, we have
$$\left|\left(1 - \frac{3}{n}\right) - 1\right| < \varepsilon.$$
Simplifying we have $|-3/n| < \varepsilon$. Since $n > 0$ we have $|-3/n| = 3/n$. Hence, we may write $3/n < \varepsilon$. Solving for n, we get $n > 3/\varepsilon$. This is our value for N. ★

With the scratch work done, the proof is straightforward.

Proof. Let $\varepsilon > 0$. Fix $N = 3/\varepsilon \in \mathbb{R}$. Let $n \in \mathbb{N}$. Assume $n > N$. Thus $n > 3/\varepsilon$. Since ε and n are positive, we get $3/n < \varepsilon$. Thus

$$|a_n - L| = \left|\left(1 - \frac{3}{n}\right) - 1\right| = \left|-\frac{3}{n}\right| = \frac{3}{n} < \varepsilon$$

as desired. □

Warning 34.14. In the previous examples, each term of the sequence is closer to the limit L than the previous term. It is tempting to think that this is a valid definition of a limit; i.e., the terms get closer and closer to L without ever actually reaching L or getting farther away. In the next example, we will see that this is not true.

Proposition 34.15. *Define a sequence $(a_n)_{n\in\mathbb{N}}$ by the formula*

$$a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

We have

$$\lim_{n\to\infty} a_n = 0.$$

This example is interesting because the sequence actually reaches the limit (every even numbered term is equal to the limit) and moves away from the limit infinitely often (each odd numbered term is farther from the limit than the previous term). However, it does *not* move too far away from the limit.

Scratch. Start with $|a_n - L| < \varepsilon$. There are two cases.

Case 1: Suppose n is odd. Then $a_n = 1/n$ so our inequality becomes $|1/n - 0| < \varepsilon$. Solving for n, as before, we reach $n > 1/\varepsilon$.

Case 2: Suppose n is even. Then $a_n = 0$ and our inequality becomes $|0 - 0| < \varepsilon$. This is true no matter the value of n, so in this case any value of N will work.

We must use an N that works in every case. Thus $N = 1/\varepsilon$ should suffice. ★

With our scratch work completed, the proof now follows.

Proof. Let $\varepsilon > 0$. Fix $N = 1/\varepsilon \in \mathbb{R}$. Let $n \in \mathbb{N}$. Assume $n > N$. We have two cases to consider.

Case 1: Suppose n is odd. In this case we have

$$|a_n - L| = |1/n - 0| = 1/n < 1/N = \varepsilon.$$

Case 2: Suppose n is even. In this case we have

$$|a_n - L| = |0 - 0| = 0 < \varepsilon.$$

Thus, in every case we have $|a_n - L| < \varepsilon$. □

34.E Divergence

Remember that in order for a sequence $a_1, a_2, a_3, \ldots$ to converge, we have:

$$\exists L \in \mathbb{R},\ \forall \varepsilon \in \mathbb{R}_{>0},\ \exists N \in \mathbb{R},\ \forall n \in \mathbb{N},\ n > N \Rightarrow |a_n - L| < \varepsilon.$$

Thus, to prove that the sequence is divergent we need to prove:

$$\forall L \in \mathbb{R},\ \exists \varepsilon \in \mathbb{R}_{>0},\ \forall N \in \mathbb{R},\ \exists n \in \mathbb{N},\ n > N \wedge |a_n - L| \geq \varepsilon.$$

As you can see, each of the four different quantifiers has been changed.

We start a divergence proof by letting $L \in \mathbb{R}$ be arbitrary. Subsequently, we must find some ε so that our sequence will continue to have some terms at least ε away from L. (Thus, ε usually depends on L.) We let $N \in \mathbb{R}$ be arbitrary, and must find a subscript n, past N, such that a_n has distance more than ε from L.

We will do one example, where ε does not depend on L.

Proposition 34.16. *Given the sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = (-1)^n$, then a_n is divergent.*

Proof. Let $L \in \mathbb{R}$. Fix $\varepsilon = 1/2 \in \mathbb{R}_{>0}$. Let $N \in \mathbb{R}$. To find a term in our sequence that is at least distance $1/2$ from L we consider two cases.

Case 1: Suppose $L \geq 0$. Fix $n \in \mathbb{N}$ to be the smallest odd integer with $n > N$. Since n is odd, $a_n = -1$. We find

$$|a_n - L| = |(-1) - L| = L + 1 > 1/2 = \varepsilon.$$

Case 2: Suppose $L < 0$. In this case we fix $n \in \mathbb{N}$ to be the smallest even integer with $n > N$. Since n is even, $a_n = 1$. We find

$$|a_n - L| = |1 - L| = 1 - L > 1/2 = \varepsilon$$

(since $L < 0$, we know $-L > 0$, hence $1 - L > 1$).

In every case, we fixed some $n \in \mathbb{N}$ satisfying both of the inequalities $n > N$ and $|a_n - L| > \varepsilon$. □

34.F One more limit trick

Sometimes when proving that a limit exists it will be useful to use one of the following functions to specify the value of N.

Definition 34.17. Let $a, b \in \mathbb{R}$. We define the *maximum* and *minimum* of these two numbers to be

$$\max(a,b) = \begin{cases} a & \text{if } a \geq b, \\ b & \text{if } a < b \end{cases} \quad \text{and} \quad \min(a,b) = \begin{cases} a & \text{if } a \leq b, \\ b & \text{if } a > b. \end{cases}$$

Example 34.18. Fix $a = 5$ and $b = 6$. Then $\max(a, b) = 6$ and $\min(a, b) = 5$. Let $x \in \mathbb{R}$, and fix $y = \max(x, 1)$ and $z = \min(x, 1)$. Then $y \geq 1$ and $z \leq 1$. △

Here is an example of a limit proof in which it is useful to specify N as the maximum of two numbers.

Proposition 34.19. *Define a sequence* $(a_n)_{n\in\mathbb{N}}$ *by the formula*

$$a_n = \frac{n+3}{2n-21}.$$

Then

$$\lim_{n\to\infty} a_n = \frac{1}{2}.$$

Scratch. Once again, we start our scratch work by considering $|a_n - L| < \varepsilon$, and try to solve for n. Thus, we want

$$\left|\frac{n+3}{2n-21} - \frac{1}{2}\right| < \varepsilon.$$

Finding a common denominator, we obtain

$$\left|\frac{2(n+3) - (2n-21)}{2(2n-21)}\right| < \varepsilon,$$

or in other words $27/|4n - 42| < \varepsilon$. Thus, we reduce to $|4n - 42| > 27/\varepsilon$.

There are two possibilities here, depending on whether $4n - 42$ is positive or negative. (Note: It is never zero since $n \in \mathbb{N}$.) In the positive case we want

$$4n - 42 > \frac{27}{\varepsilon},$$

which reduces (after some algebra) to

$$n > \frac{27 + 42\varepsilon}{4\varepsilon}. \tag{34.20}$$

In the case when $4n - 42$ is negative, we want

$$42 - 4n > \frac{27}{\varepsilon},$$

which reduces to

$$n < \frac{42\varepsilon - 27}{4\varepsilon}. \tag{34.21}$$

Inequality (34.21) does not help us since we want to find a *lower* bound on n. Thus, we must guarantee that case never happens, or in other words, we must guarantee

that $4n - 42$ is positive. This means we want to take $4n - 42 > 0$, or in other words $n > 21/2$. Combining this with (34.20), we want to take

$$N = \max\left(\frac{21}{2}, \frac{27 + 42\varepsilon}{4\varepsilon}\right),$$

so that if $n > N$ then n is greater than both $21/2$ and $(27 + 42\varepsilon)/(4\varepsilon)$. ★

We are now ready to begin our proof.

Proof. Let $\varepsilon > 0$. Fix $N = \max(21/2, (27 + 42\varepsilon)/4\varepsilon) \in \mathbb{R}$. Let $n \in \mathbb{N}$. Assume $n > N$. Using the fact that $N \geq 21/2$, we have $4n - 42 > 4N - 42 \geq 0$. We also find

$$4n - 42 > 4N - 42 \geq 4\frac{27 + 42\varepsilon}{4\varepsilon} - 42 = \frac{27}{\varepsilon}.$$

Hence, we find that

$$|a_n - L| = \left|\frac{n+3}{2n-21} - \frac{1}{2}\right| = \left|\frac{27}{4n-42}\right| = \frac{27}{4n-42} < \frac{27}{27/\varepsilon} = \varepsilon.$$ □

The trick used in the proof above is that we can use maximums to guarantee our sequence is far enough along so that it behaves in a certain way (in this case, we needed it to stay positive).

34.G Exercises

Exercise 34.1. Write the first six terms, and determine the nth term a_n, for each of the following sequences.

(a) An arithmetic sequence with first term 5 and common difference -3.
(b) A geometric sequence with first term 4 and common ratio 2.
(c) An arithmetic sequence with first term $1/2$ and common difference $3/4$.
(d) A geometric sequence with first term $3/5$ and common ratio $2/3$.

Exercise 34.2. Translate the following phrases into symbolic logic. (Your answer should include any necessary quantifications on the variables ε, N, and n.)

(a) The sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = 3 - 4/n$ converges to $L = 3$.
(b) The sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = 6$ does not converge to $L = 3$. (Note: This sequence does converge to $L = 6$.)

Exercise 34.3. Let $a, b, x \in \mathbb{R}$. Prove the following facts.

(a) $\max(a, b) \geq a$ and $\max(a, b) \geq b$.
(b) $\min(a, b) \leq a$ and $\min(a, b) \leq b$.
(c) If $x > \max(a, b)$, then $x > a$ and $x > b$.

Exercise 34.4. Prove that

$$\lim_{n\to\infty} \frac{2}{n^2} = 0.$$

Exercise 34.5. Prove that

$$\lim_{n\to\infty} \frac{3n-5}{2n+4} = \frac{3}{2}.$$

(Hint: When $n \in \mathbb{N}$, then $2n+4$ is always positive, so you don't have to worry about when it is negative.)

Exercise 34.6. Prove or disprove: The sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = (n+1)/n$ converges. (Hint: On scratch paper, write out the first ten terms to see if the sequence is going somewhere.)

Exercise 34.7. Let $c \in \mathbb{R}$. Assume that $(a_n)_{n\in\mathbb{N}}$ is the arithmetic sequence with first term c and common difference 0. Prove that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to c. (In other words, prove that the constant sequence $c, c, c, \ldots$ converges to c.)

Exercise 34.8. Prove that the sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = n$ does not converge to $L = 3$.

Exercise 34.9. Prove that $\lim_{n\to\infty}(\sqrt{n^2+1} - n) = 0$.

Exercise 34.10. Assume that $(a_n)_{n\in\mathbb{N}}$ is the geometric sequence with first term c and common ratio r, for some $c, r \in \mathbb{R}$. Prove the following statements.

(a) If $|r| < 1$, then a_n converges to 0.
(b) If $c \neq 0$ and a_n converges to 0, then $|r| < 1$.
(c) If $c > 0$ and $r > 1$, then a_n diverges.

(Feel free to use laws of logarithms as well as the fact that if $a, b \in \mathbb{R}$ with $0 < a < b$, then $\ln(a) < \ln(b)$. In particular, when $0 < r < 1$ we have $\ln(r) < 0$, and when $r > 1$ we have $\ln(r) > 0$.)

35 Series

In this section we introduce the idea of adding infinitely many objects together. There are many applications for these ideas, which lead naturally into the development of the integral in calculus.

35.A What is a series?

We write $\sum_{n=1}^{\infty} a_n$ to denote the *series* $a_1 + a_2 + a_3 + \cdots$. But what does this really mean?

The numbers a_n form a sequence, which we call the *terms* or the *summands* of the series. From those terms, we can form an entirely *new* sequence with terms

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ s_4 &= a_1 + a_2 + a_3 + a_4, \\ &\vdots \end{aligned}$$

which is called the sequence of *partial sums*. These sums are only the beginning portion of the series, which is why we call them partial sums. We can also define the partial sums using the more compact formula

$$s_{n+1} = s_n + a_{n+1}, \tag{35.1}$$

which we sometimes will find useful.

We say that the series $\sum_{n=1}^{\infty} a_n$ *converges* to a sum S if $\lim_{n\to\infty} s_n = S$. In other words, the series converges when the sequence of partial sums converges. Thus, the methods of the previous section apply.

When first working with series it is important to note that the sequence of terms $a_1, a_2, a_3, \ldots$ is very different from the sequence of partial sums $s_1, s_2, s_3, \ldots$. For instance, it is possible that the terms converge but the partial sums diverge.

In the next two examples, we will investigate how to describe the partial sums when given a sequence of summands.

Example 35.2. Define $(a_n)_{n\in\mathbb{N}}$ by the rule $a_n = 1$. In other words, $(a_n)_{n\in\mathbb{N}}$ is the constant sequence of 1's. We find that the partial sums for this sequence are

$$\begin{aligned} s_1 &= a_1 = 1, \\ s_2 &= a_1 + a_2 = 1 + 1 = 2, \\ s_3 &= a_1 + a_2 + a_3 = 1 + 1 + 1 = 3, \\ s_4 &= a_1 + a_2 + a_3 + a_4 = 1 + 1 + 1 + 1 = 4, \\ &\vdots \end{aligned}$$

and we would guess that, in general, $s_n = n$. We will prove that our guess is correct, by induction.

Proof. **Base case:** When $n = 1$, we have $s_1 = 1$.

Inductive step: Assume $s_k = k$ for some integer $k \geq 1$. We want to show $s_{k+1} = k + 1$. Using (35.1), we have

$$s_{k+1} = s_k + a_{k+1} = s_k + 1 = k + 1,$$

since $s_k = k$ by the inductive hypothesis. □

In this example the sequence of terms $a_1, a_2, a_3, \ldots$ converges to 1. However, the sequence of partial sums $s_1, s_2, s_3, \ldots$ diverges. △

Example 35.3. Consider the sequence $(b_n)_{n\in\mathbb{N}}$ defined by the rule

$$b_n = \frac{1}{n} - \frac{1}{n+1}.$$

The first few terms in this sequence are given by

$$b_1 = \frac{1}{1} - \frac{1}{2} = \frac{1}{2},$$
$$b_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$
$$b_3 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}, \quad \text{and}$$
$$b_4 = \frac{1}{4} - \frac{1}{5} = \frac{1}{20},$$

and it appears that this sequence is converging (fairly quickly) to 0.

Now consider the sequence of partial sums. The first few terms are

$$s_1 = b_1 = \frac{1}{2},$$
$$s_2 = b_1 + b_2 = s_1 + b_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$
$$s_3 = b_1 + b_2 + b_3 = s_2 + b_3 = \frac{2}{3} + \frac{1}{12} = \frac{3}{4},$$
$$s_4 = b_1 + b_2 + b_3 + b_4 = s_3 + b_4 = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}, \quad \text{and}$$
$$s_5 = b_1 + b_2 + b_3 + b_4 + b_5 = s_4 + b_5 = \frac{4}{5} + \frac{1}{30} = \frac{5}{6}.$$

They appear to be converging to 1. Before we can prove convergence, we need more information about the sequence of partial sums.

It appears that the partial sums are given by the rule $s_n = 1 - \frac{1}{n+1}$. (Try it for the first four terms above.) We will now prove that this is correct, by induction.

Proof. **Base case:** When $n = 1$, then $s_1 = 1/2 = 1 - 1/2$.

Inductive step: Assume $s_k = 1 - 1/(k+1)$ for some integer $k \geq 1$. We want to show $s_{k+1} = 1 - 1/(k+2)$. Using (35.1), we have

$$s_{k+1} = s_k + b_{k+1} = \left(1 - \frac{1}{k+1}\right) + \left(\frac{1}{k+1} - \frac{1}{k+2}\right) = 1 - \frac{1}{k+2}$$

as desired. □

Now that we know $s_n = 1 - 1/(n+1)$, we can prove $\lim_{n\to\infty} s_n = 1$. We will not include the scratch work, but here is the proof.

Proof. Let $\varepsilon > 0$. Fix $N = \frac{1}{\varepsilon} - 1 \in \mathbb{R}$. Let $n \in \mathbb{N}$. Assume $n > N$. We find

$$|s_n - S| = \left|1 - \frac{1}{n+1} - 1\right| = \frac{1}{n+1} < \frac{1}{N+1} = \varepsilon$$

as desired. □

This finishes our proof that $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$. Those who have taken calculus might recognize that this is a telescoping sum, which makes it much easier to simplify. △

Advice 35.4. To prove that a series converges try the following steps:
(1) Compute a few partial sums.
(2) Conjecture a general formula for the partial sums.
(3) Prove that formula by induction.
(4) Using your formula for the partial sums, find the limit.
(5) Finally, prove that the partial sums converge to that limit.

We will demonstrate how to prove convergence of series with one more example, leaving most of the work as Exercise 35.3.

Example 35.5. Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. The first few partial sums work out to be $1/2, 3/4, 7/8, 15/16, \ldots$. By induction, we can prove that

$$s_n = 1 - \frac{1}{2^n}.$$

These partial sums converge to 1. △

We now prove a powerful result that can often be used to show that a series does not converge. We first state and prove the result in terms of convergence, and give the contrapositive (in terms of divergence) as a corollary. Before reading this proof, it might be helpful to review the triangle inequality (Theorem 8.21).

Theorem 35.6. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. If the series*

$$\sum_{i=1}^{\infty} a_i$$

converges, then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. We will write

$$s_n = \sum_{i=1}^{n} a_i$$

for the partial sums of the series. The theorem asserts that if $\lim_{n\to\infty} s_n$ exists, then $\lim_{n\to\infty} a_n = 0$. Note that for any integer $n \geq 2$, we have $s_n - s_{n-1} = a_n$.

Assume that the series converges; in other words, assume that the sequence $(s_n)_{n\in\mathbb{N}}$ of partial sums converges to some limit L. Our goal is now to show that

$$\lim_{n\to\infty} a_n = 0.$$

Let $\varepsilon > 0$ be arbitrary. We wish to find an $N \in \mathbb{R}$ such that for any natural number $n > N$, we have $|a_n| < \varepsilon$.

Since $\varepsilon > 0$, we also have that $\varepsilon/2 > 0$. Hence, since the sequence $(s_n)_{n\in\mathbb{N}}$ converges, there is some $M \in \mathbb{R}$ such that for any natural number $n > M$, we have $|s_n - L| < \varepsilon/2$. Taking $N = M + 1$, and letting n be a natural number with $n > N$, then both n and $n - 1$ are greater than M. Hence,

$$|s_n - L| < \varepsilon/2 \quad \text{and} \quad |s_{n-1} - L| < \varepsilon/2.$$

Therefore,

$$\begin{aligned}
|a_n| &= |s_n - s_{n-1}| \\
&= |(s_n - L) + (L - s_{n-1})| \\
&\leq |s_n - L| + |L - s_{n-1}| && \text{(by the triangle inequality)} \\
&= |s_n - L| + |s_{n-1} - L| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

We thus see that for this value of N, it is true that for any natural number $n > N$ we have $|a_n - 0| < \varepsilon$. Therefore

$$\lim_{n\to\infty} a_n = 0.$$

□

Corollary 35.7. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. If*

$$\lim_{n\to\infty} a_n \neq 0$$

then

$$\sum_{i=1}^{\infty} a_i$$

does not converge.

Example 35.8. The series

$$\sum_{i=1}^{\infty} \frac{i+3}{2i-21}$$

does not converge, because

$$\lim_{n\to\infty} \frac{n+3}{2n-21} = \frac{1}{2} \neq 0$$

by Proposition 34.19. △

Example 35.9. The series

$$\sum_{i=1}^{\infty} (-1)^i$$

does not converge, since

$$\lim_{n\to\infty} (-1)^n$$

does not exist (see Proposition 34.16), and hence does not equal 0. △

Note that the converse of Theorem 35.6 does not hold; it is possible for

$$\lim_{n\to\infty} a_n = 0$$

to hold while

$$\sum_{i=1}^{\infty} a_i$$

does not converge. See Exercise 35.5 for an example.

35.B Exercises

Exercise 35.1. Consider the sequence $(a_n)_{n\in\mathbb{N}}$ given by the rule $a_n = n$. Find the first 6 terms of the sequence of partial sums $(s_n)_{n\in\mathbb{N}}$. Conjecture a simple formula for s_n and prove it.

Exercise 35.2. Let $c, d \in \mathbb{R}$ and fix $(a_n)_{n\in\mathbb{N}}$ to be the arithmetic sequence defined by $a_n = c + (n-1)d$. (This is the arithmetic sequence with first term c and common difference d.) Find a formula for the nth partial sum $s_n = \sum_{k=1}^{n} a_k$ and prove it.

Exercise 35.3. Give a complete proof that

$$\sum_{n=1}^{\infty} 1/2^n = 1,$$

by filling in the missing details from Example 35.5. This should include:
(a) Giving a proof (by induction) that $s_n = 1 - 1/2^n$.
(b) Giving a proof that $\lim_{n\to\infty} s_n = 1$.

Exercise 35.4. Prove or disprove: The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges.

Exercise 35.5. In this exercise we will show that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge. Throughout the exercise, for each integer $n \geq 1$ take

$$s_n = \sum_{k=1}^{n} \frac{1}{k},$$

which is the nth partial sum.

(a) The main idea for showing that the harmonic series diverges is to break the series into pieces

$$1 + \underbrace{\boxed{\frac{1}{2}}}_{t_1} + \underbrace{\boxed{\frac{1}{3} + \frac{1}{4}}}_{t_2} + \underbrace{\boxed{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}}_{t_3} + \underbrace{\boxed{\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{15} + \frac{1}{16}}}_{t_4} + \cdots,$$

where each box contains twice as many terms as the previous box.
Notice that

$$t_1 = \frac{1}{2} \geq \frac{1}{2},$$

$$t_2 = \frac{1}{3} + \frac{1}{4} \geq \frac{1}{4} + \frac{1}{4} = 2 \cdot \frac{1}{4} = \frac{1}{2}, \text{ and}$$

$$t_3 = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 4 \cdot \frac{1}{8} = \frac{1}{2},$$

where the inequalities come from replacing terms with possibly smaller fractions.
In general, for each integer $n \geq 1$ define

$$t_n = \sum_{k=2^{n-1}+1}^{2^{n-1}+2^{n-1}} \frac{1}{k} = \sum_{k=2^{n-1}+1}^{2^n} \frac{1}{k} = \frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}.$$

Prove that $t_n \geq \frac{1}{2}$ for each integer $n \geq 1$. (Hint: How many terms are being added? What is the smallest one?)

(b) Show that $s_{2^n} \geq 1 + \frac{n}{2}$, for each integer $n \geq 0$, by induction.
(Hint: For each integer $n \geq 0$, we have $s_{2^{n+1}} = s_{2^n} + t_{n+1}$.)

(c) Now show that the harmonic series does not converge.

36 Limits of functions

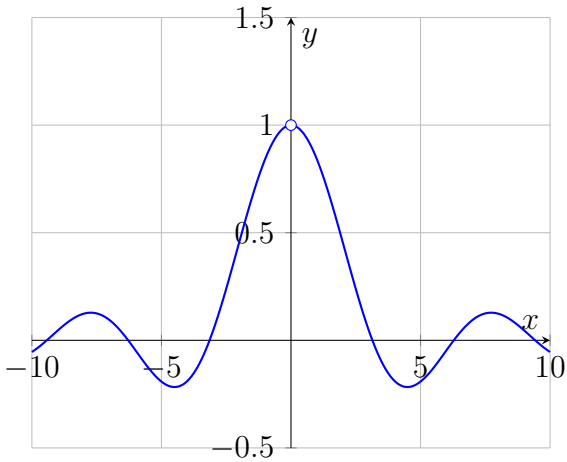
Define $f\colon \mathbb{R} - \{0\} \to \mathbb{R}$ by the rule

$$f(x) = \frac{\sin x}{x}.$$

If we evaluate this function for some values of x near 0, we find the following interesting behavior.

x	0.1	0.01	0.001
$f(x)$	0.998334	0.999983	0.999999

It appears that as x gets close to 0, the value of $f(x)$ gets close to 1. (Note that we cannot just plug $x = 0$ into the function, since that would result in division by 0.) The graph of the function $f(x)$ seems to confirm this behavior.



However, graphs can sometimes be misleading. Thus, in this section we will formalize what we mean when we say that a function approaches a value. This will become our definition of the limit of a function.

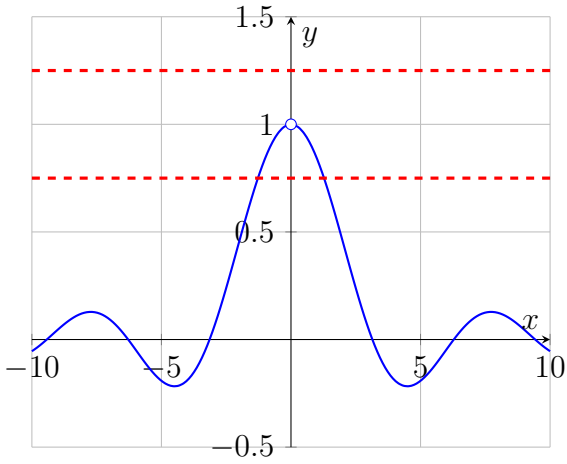
36.A Windows

In the example above, there are two related quantities we focus upon: the input x of the function, and the output $f(x)$. We are interested in the behavior of the outputs $f(x)$ as x approaches some *fixed* constant, which we might call the *point of interest*. In the previous example that point of interest is 0, but more generally we use the letter $a \in \mathbb{R}$ to describe the place where x is headed. Sometimes we write $x \to a$ as shorthand for the sentence "as x goes towards the point of interest a."

Similarly, we use the letter L to denote the limiting value that $f(x)$ approaches (if any) as $x \to a$. We might write $f(x) \to L$ to mean that "$f(x)$ is approaching the limit L."

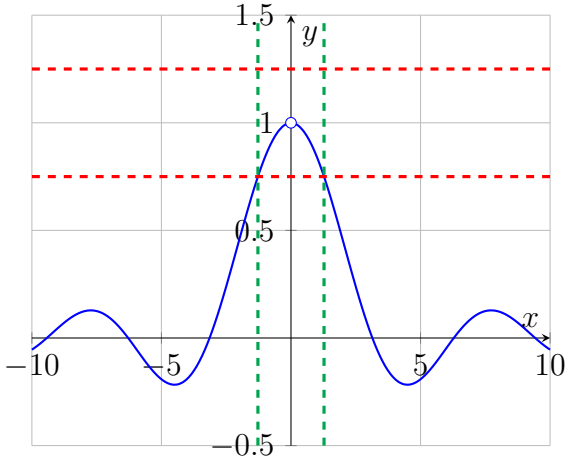
To formalize all of this, we start (just as with sequences) by letting $\varepsilon > 0$ denote some (positive, arbitrarily small) quantity that tells us how close our function should

be to the limit L. For instance, in the example above, when $\varepsilon = 1/4$ we want our function to stay between the two dashed lines in the graph below.



The top dashed line is $y = L + \varepsilon$ and the bottom dashed line is $y = L - \varepsilon$. We want the graph of $y = f(x)$ to stay between these two lines, but of course, as you can plainly see, the function does *not* stay between those two lines everywhere. However, it does stay between those two lines near the point of interest $a = 0$.

The next question we must answer is: How close must the input x get to the point of interest a so that the output $f(x)$ stays between those two lines? For example, let's add two (green) vertical dashed lines to our previous graph, which tell us exactly how far away from $a = 0$ the values of x can range before the function takes on values outside the band bordered by the (red) horizontal dashed lines.



Those vertical lines are (approximately) $x = 1.275698$ and $x = -1.275698$. Thus we see that x can vary as much as 1.275698 away from the point of interest, and our function stays within $\varepsilon = 1/4$ of the limit. If we take a smaller value of ε, then we have less room to vary.

Those two horizontal lines along with the two vertical lines give us a *window* (the rectangle in the middle). As we take smaller values of ε, that window should squeeze

in on the limit L. Notice that if the graph of our function has points between the two green lines that are also either above the top red line or below the bottom red line (i.e., directly above or below the window), then we have not placed the vertical lines correctly.

We let $\delta > 0$ be a variable that measures (given some ε) how far x can vary away from the point of interest and still guarantee that $f(x)$ stays within a distance of ε from the limit value L. In other words, δ is some constant small enough that the vertical lines $x = a + \delta$ and $x = a - \delta$ together with the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ produce a window for our function.

36.B Limit definition

We begin by defining certain subsets of $\mathbb{R}$.

Definition 36.1. A *deleted neighborhood* of $a \in \mathbb{R}$ is a subset of $\mathbb{R}$ of the form $(c, d) - \{a\}$, where $c < d$ are real, and $a \in (c, d) \subseteq \mathbb{R}$.

We can think of a deleted neighborhood as a set of real numbers that contains all the points "close to" a (but does not contain a itself).

Example 36.2. Fix $a = 2$. Some deleted neighborhoods of a include

$$(-10, 2.1) - \{2\}, \quad (1, 3) - \{2\}, \quad (1.9999999, 2.00000001) - \{2\}. \qquad \triangle$$

We are now equipped to give the formal definition of what we mean by a limit for functions. Afterwards, we will explain all of the notation.

Definition 36.3. Suppose we are given a point of interest $a \in \mathbb{R}$, and a possible limit value $L \in \mathbb{R}$, along with a function $f \colon S \to \mathbb{R}$, where S is a subset of $\mathbb{R}$ that includes some deleted neighborhood of a. We write

$$\lim_{x \to a} f(x) = L$$

to mean that

$$\forall \varepsilon \in \mathbb{R}_{>0},\ \exists \delta \in \mathbb{R}_{>0},\ \forall x \in S,\ 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

In this case we say that the *limit* of the function f, as x approaches the point of interest a, is the real number L.

What is ε? As before, ε measures how close the function gets to the limit L. It is allowed to get arbitrarily small.

What is δ? The second quantifier is "$\exists \delta \in \mathbb{R}_{>0}$". The variable δ measures how close x must be to the point a in order for our function to stay within ε of the limit. It must be fixed in terms of ε.

What is S? The domain of the function f is S; it is the set of real numbers on which f is defined. We want $f(x)$ to be defined for any real number x that is "close to" a, but not necessarily at a. We do this by requiring that some deleted neighborhood of a is a subset of S. This reflects the idea that a limit depends on what happens near (but not at) a.

What is x? It is the variable we are plugging into our function f.

What does the premise of the implication, $0 < |x-a| < \delta$, say? It says two different things. First, the inequality $0 < |x - a|$ tells us that $x \neq a$. When defining limits, we do not care what $f(x)$ actually does at $x = a$, only what it does nearby. Second, the inequality $|x - a| < \delta$ tells us that x is allowed to range only a distance of δ away from the point of interest.

What does the conclusion of the implication, $|f(x) - L| < \varepsilon$, say? This condition says that the function value $f(x)$ is within a distance of ε from the limit L. Equivalently, by removing the absolute value signs, we may write $-\varepsilon < f(x) - L < \varepsilon$, or in other words, $L - \varepsilon < f(x) < L + \varepsilon$.

When handling limits, each proof should look nearly the same. We give the general outline of such a proof below. The reader should note how each quantifier is dealt with in turn, and the implication will be proved directly.

Proof outline.
Let $\varepsilon > 0$.
Fix $\delta = \boxed{\text{found in scratch work}} > 0$.
Let $x \in S$.
Assume $0 < |x - a| < \delta$.
Reverse the steps in scratch work.
Conclude that $|f(x) - L| < \varepsilon$. □

36.C Examples of limits

Some of the easiest limits are for linear functions.

Example 36.4. Fix $f(x) = 3x + 2$, fix $a = 2$, and fix $L = 8$. (Note: If we plug $x = 2$ into f, we get $f(2) = 8$. In this case we do believe that the limit will also equal the function's value at the point of interest.) We will show that

$$\lim_{x \to 2} 3x + 2 = 8.$$

Scratch. Since this is the scratch work, we start with the conclusion $|f(x) - L| < \varepsilon$, and try to eventually get information about the quantity $|x - a| = |x - 2|$. We have

$$|f(x) - L| = |(3x + 2) - 8| = |3x - 6| = 3|x - 2|.$$

Thus we want $3|x - 2| < \varepsilon$, or in other words $|x - 2| < \varepsilon/3$. This tells us what value for δ we should use. ★

With the scratch work done, we can now give the formal proof.

Proof. Let $\varepsilon > 0$. Fix $\delta = \varepsilon/3 > 0$. Let $x \in \mathbb{R}$. Assume $0 < |x-2| < \delta$. We then find

$$|f(x) - L| = |(3x+2) - 8| = |3x - 6| = 3|x-2| < 3\delta = 3(\varepsilon/3) = \varepsilon$$

as desired. □

Limits for other linear polynomials will work similarly. △

Example 36.5. We find $L \in \mathbb{R}$ such that $\lim_{x\to -1} -2x + 3 = L$.

Set $f(x) = -2x + 3$. Plugging in the point of interest $a = -1$, we find that $f(-1) = (-2)(-1) + 3 = 5$. Thus we would guess the limit is $L = 5$. We now prove it (without including our scratch work).

Proof. Let $\varepsilon > 0$. Fix $\delta = \varepsilon/2 > 0$. Let $x \in \mathbb{R}$. Assume $0 < |x - (-1)| < \delta$. We find

$$|f(x) - L| = |(-2x+3) - 5| = |-2x - 2| = 2|x+1| < 2\delta = 2(\varepsilon/2) = \varepsilon$$

as desired. □

When working with a linear function $f(x) = cx + d$ (for some constants $c, d \in \mathbb{R}$ with $c \neq 0$) the best value for δ should be $\delta = \varepsilon/|c|$. In this example we have $c = -2$ and indeed, the value for δ was $\varepsilon/|c|$. △

There is one last trick that will help us to evaluate limits. In limit proofs we have *one* assumption, namely

$$0 < |x - a| < \delta.$$

However, we also have some control on which δ we fix. If we guarantee that $\delta \leq 1$, then our assumption gives us $|x - a| < 1$, or in other words

$$a - 1 < x < a + 1.$$

This allows us to limit x a lot. (If we need an even smaller interval containing x, we can take δ even smaller.) We will use this idea to find the limit of a quadratic function.

Proposition 36.6. $\lim_{x\to 1} x^2 + x = 2$.

We will include the scratch work, so you can see how this is done. (Normally, it should not appear in your proof.)

Scratch. Start with $|f(x) - L| < \varepsilon$. We have $f(x) = x^2 + x$ and $L = 2$, and so plugging in those values we want $|x^2 + x - 2| < \varepsilon$. If we factor the left side of the inequality, we get

$$|x - 1| \cdot |x + 2| < \varepsilon. \tag{36.7}$$

At this point, one might want to take $\delta = \varepsilon/|x+2|$, but δ must not depend on x. (Why?) To handle this issue, we bound $|x+2|$ as follows. Assuming $\delta \leq 1$, we get $|x-1| < \delta \leq 1$, and hence $-1 < x - 1 < 1$. Thus, by adding 3 throughout, we have $2 < x + 2 < 4$. Now $2 < |x+2| < 4$. Using this bound in (36.7), we see that we need $|x-1| < \varepsilon/4$. ★

We are now ready for the proof.

Proof. Let $\varepsilon > 0$. Fix $\delta = \min(1, \varepsilon/4) > 0$. Let $x \in \mathbb{R}$. Assume that we have $0 < |x - 1| < \delta$. First, since $\delta \leq 1$, this tells us $|x - 1| < 1$, so $-1 < x - 1 < 1$. Therefore $2 < x + 2 < 4$, and taking absolute values we get $|x + 2| < 4$. Second, since $\delta \leq \varepsilon/4$ we find

$$|f(x) - L| = |(x^2 + x) - 2| = |x - 1||x + 2| < \delta \cdot 4 \leq (\varepsilon/4)4 = \varepsilon$$

as desired. □

We end this section with one final example of a limit proof.

Proposition 36.8.

$$\lim_{x \to 2} \frac{2x + 1}{3x + 2} = \frac{5}{8}.$$

Before reading the proof below, try to do the scratch work yourself, and see if it helps you figure out the choices made in the proof.

Proof. Let $\varepsilon > 0$. Fix $\delta = \min(1, 40\varepsilon) > 0$. Let $x \in \mathbb{R} - \{-2/3\}$. (Notice that we have to avoid $x = -2/3$ since the function is not defined there.) Assume $0 < |x - 2| < \delta$. First, since $\delta \leq 1$, this tells us $|x - 2| < 1$ hence $-1 < x - 2 < 1$. Adding 2 we get $1 < x < 3$. Multiplying by 3 and adding 2, we get $5 < 3x + 2 < 11$. Taking reciprocals, we find that $1/11 < 1/(3x + 2) < 1/5$. Then taking absolute values, we have $1/|3x + 2| < 1/5$. Second, since $\delta \leq 40\varepsilon$ we find

$$|f(x) - L| = \left|\frac{2x + 1}{3x + 2} - \frac{5}{8}\right| = \left|\frac{x - 2}{8(3x + 2)}\right| = \frac{|x - 2|}{8 \cdot |3x + 2|} < \frac{\delta}{8 \cdot 5} \leq \frac{40\varepsilon}{40} = \varepsilon. \qquad \square$$

Advice 36.9. When simplifying $|f(x) - L|$ you should expect that $|x - a|$ is one of the factors. That can help you simplify the expression (and also gives a quick double-check that you have not made an algebra mistake).

36.D Exercises

Exercise 36.1. Prove that

$$\lim_{x \to 4} 2x + 3 = 11.$$

Exercise 36.2. Let $a, c, d \in \mathbb{R}$. Prove that

$$\lim_{x \to a} cx + d = ca + d.$$

(Hint: Consider the cases in which $c = 0$ and $c \neq 0$, separately. Be sure that your proof properly handles the case where $c < 0$.)

Exercise 36.3. Prove that

$$\lim_{x\to 5} x^2 + 3x + 3 = 43.$$

Exercise 36.4. Prove that

$$\lim_{x\to 2} \frac{7x+4}{4x+1} = 2.$$

Exercise 36.5. Prove that

$$\lim_{x\to 3} x^3 + x^2 + 2 = 38.$$

37 Continuity

37.A Defining continuity

Intuitively, continuity for a function $f\colon \mathbb{R} \to \mathbb{R}$ means that there are no holes or jumps in the function. Put another way, if we focus on a point of interest $a \in \mathbb{R}$, we need $f(a)$ to be defined, and for any x near a we want $f(x)$ to be near $f(a)$. Thus, there are two separate conditions that combine to give the formal definition of continuity.

Definition 37.1. Let $S \subseteq \mathbb{R}$. Given a function $f\colon S \to \mathbb{R}$ and a point of interest $a \in S$, we say that f is *continuous at* a if
(1) S includes an open interval around a and
(2) $\lim_{x\to a} f(x) = f(a)$.
If a function is continuous at all points in its domain, we say that the function is *continuous* (everywhere that it is defined).

Example 37.2. Define $f\colon \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = cx + d$, where $c, d \in \mathbb{R}$. By Exercise 36.2, for any $a \in \mathbb{R}$ we have

$$\lim_{x\to a} f(x) = ca + d = f(a).$$

Hence, f is continuous at a for each $a \in \mathbb{R}$. Since f is continuous at all points in its domain, we can thus say that f is continuous. △

Example 37.3. Fix $f\colon \mathbb{R} \to \mathbb{R}$ to be the characteristic function of the set $[0, 1]$. Then

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

One can check that f is continuous at each $a \in \mathbb{R} - \{0, 1\}$. We will show that f is not continuous at 1. (The other numbers in the domain are handled in Exercise 37.3.)

Note that $f(1) = 1$. Then we need to show that

$$\lim_{x\to 1} f(x) = 1$$

is false. Hence, we need to show that

$$\exists \varepsilon > 0, \forall \delta > 0, \exists x \in \mathbb{R},\ (0 < |x - 1| < \delta) \wedge (|f(x) - 1| \ge \varepsilon).$$

Choose $\varepsilon = 1/2$. Let $\delta > 0$ be arbitrary, and fix $x = 1 + \delta/2$. We notice that $0 < |x - 1| = \delta/2 < \delta$ and, since $x > 1$,

$$|f(x) - 1| = |0 - 1| = 1 > \varepsilon.$$

Hence, f is not continuous at 1. △

Example 37.4. Define $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ by the rule $f(x) = x$. Note that 0 is in the domain of f, but the domain of f does not include any open interval around 0. Hence, f is not continuous at 0. Since 0 belongs to the domain of f, we see that f is a discontinuous function. (Some texts would say this function is continuous at 0, using an alternative definition of continuity for endpoints of intervals. This illustrates that one must carefully check definitions when verifying statements.) △

37.B Building new functions from old

We will now discuss several different ways to combine functions to form new functions.

Definition 37.5. Let $A, B \subseteq \mathbb{R}$, and let $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ be functions. We define a new function $f + g\colon A \cap B \to \mathbb{R}$ by the rule

$$(f + g)(x) = f(x) + g(x).$$

We may also define a new function $fg\colon A \cap B \to \mathbb{R}$ by

$$(fg)(x) = f(x)g(x).$$

If we fix $B' = \{x \in B \,:\, g(x) \neq 0\}$, we may define a function $f/g\colon A \cap B' \to \mathbb{R}$ by

$$(f/g)(x) = f(x)/g(x).$$

Example 37.6. Fix $f(x) = x^2$ and fix $g(x) = 2x + 1$, which are functions defined on all real numbers. Then $(f + g)(x) = x^2 + 2x + 1$ and $(fg)(x) = x^2(2x + 1)$ are functions defined on all real numbers. Note that $g(x) = 0$ when $x = -1/2$. Hence, $(f/g)(x) = x^2/(2x + 1)$ is a function defined on $\mathbb{R} - \{-1/2\}$. △

37.C Limit laws

When we build a function by adding or multiplying two known functions, it turns out that we can prove that the limits of the functions interact nicely.

Theorem 37.7. *Let f and g be real-valued functions defined on a deleted neighborhood S of $a \in \mathbb{R}$. Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then*

$$\lim_{x\to a}(f + g)(x) = L + M$$

Proof. Assume that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. We wish to show that $\lim_{x\to a}(f + g)(x) = L + M$. We will do this by choosing an arbitrary ε and finding a δ such that for any $x \in S$,

$$0 < |x - a| < \delta \Rightarrow |(f + g)(x) - (L + M)| < \varepsilon.$$

Let $\varepsilon > 0$. Then $\varepsilon/2 > 0$. Hence, since $\lim_{x\to a} f(x) = L$, there is some $\delta_1 > 0$ such that for any $x \in S$, we have the implication

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon/2. \tag{37.8}$$

Similarly, since $\lim_{x\to a} g(x) = M$, there is some $\delta_2 > 0$ such that for any $x \in S$, we have

$$0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon/2. \tag{37.9}$$

Choose $\delta = \min(\delta_1, \delta_2)$. Let $x \in S$, and assume $0 < |x - a| < \delta$. Since $\delta \leq \delta_1$, and $\delta \leq \delta_2$, from (37.8) and (37.9) we get

$$|f(x) - L| < \varepsilon/2 \text{ and } |g(x) - M| < \varepsilon/2.$$

Now, using the triangle inequality we have

$$\begin{aligned} |(f + g)(x) - (L + M)| &= |f(x) + g(x) - L - M| \\ &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Hence,

$$\lim_{x\to a}(f + g)(x) = L + M. \qquad \square$$

Theorem 37.10. *Let f and g be real-valued functions defined on a deleted neighborhood S of $a \in \mathbb{R}$. Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then*

$$\lim_{x\to a}(fg)(x) = LM.$$

Proof. We first do the proof under the assumption that one of L and M is not zero. Without loss of generality, we assume that $L \neq 0$.

Let $\varepsilon > 0$. Then $\varepsilon/(2|L|) > 0$, so, since $\lim_{x\to a} g(x) = M$, we can find a $\delta_1 > 0$ so that for any $x \in S$, if $0 < |x - a| < \delta_1$, we have $|g(x) - M| < \varepsilon/(2|L|)$. Note that this conclusion further implies that $|g(x)| < |M| + \varepsilon/(2|L|)$.

Now

$$\frac{\varepsilon}{2\left(\varepsilon/(2|L|) + |M|\right)} > 0,$$

so there is some δ_2 so that when $0 < |x - a| < \delta_2$, we have

$$|f(x) - L| < \frac{\varepsilon}{2\left(\varepsilon/(2|L|) + |M|\right)}.$$

Choose $\delta = \min(\delta_1, \delta_2)$. Let $x \in S$.

Now, assume that $0 < |x - a| < \delta$. Then $|x - a| < \delta_1$ and $|x - a| < \delta_2$. Hence, $|g(x) - M| < \varepsilon/(2|L|)$ and

$$|f(x) - L| < \frac{\varepsilon}{2\left(\varepsilon/(2|L|) + |M|\right)}.$$

We then have

$$\begin{aligned}
|(fg)(x) - LM| &= |f(x)g(x) - LM| \\
&= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&= |(f(x) - L)g(x) + L(g(x) - M)| \\
&\leq |(f(x) - L)g(x)| + |L(g(x) - M)| \\
&= |f(x) - L||g(x)| + |L||g(x) - M| \\
&< \frac{\varepsilon}{2\left(\varepsilon/(2|L|) + |M|\right)}(|M| + \varepsilon/(2|L|)) + |L|\varepsilon/(2|L|) \\
&= \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

Hence,

$$\lim_{x \to a}(fg)(x) = LM.$$

The proof in the case that both L and M are equal to 0 is much less complicated, and is left to the reader (see Exercise 37.4). □

One can also prove that the limit of f/g is the limit of f over the limit of g, provided that the limit of g is nonzero. We state the theorem here, without proof.

Theorem 37.11. *Let f and g be real-valued functions defined on a deleted neighborhood S of $a \in \mathbb{R}$. Suppose that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ with $M \neq 0$. Then*

$$\lim_{x \to a}(f/g)(x) = L/M.$$

From the previous three theorems, we can deduce the following result about continuity.

Theorem 37.12. *Let S be a subset of $\mathbb{R}$, and let $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ be real-valued functions that are continuous at some $a \in S$. Then $f + g$, and fg are continuous at a. If $g(a) \neq 0$, then f/g is continuous at a.*

Proof. We prove the theorem for the sum; the proofs for the product and the quotient are similar.

Suppose that f and g are continuous at a. Then

$$\lim_{x \to a}(f + g)(x) = \left(\lim_{x \to a} f(x)\right) + \left(\lim_{x \to a} g(x)\right) = f(a) + g(a) = (f + g)(a).$$

Hence, $f + g$ is continuous at a. □

Remark 37.13. Note that if f and g are continuous at all points in their domain, then so is their sum and their product. The quotient f/g will be continuous at all points of S at which g is nonzero. ▲

37.D Continuity of polynomials

In Exercise 36.2, you were asked to prove that any linear function (that is, a function of the form $f(x) = cx + d$) is continuous at all real numbers. This implies that constant functions are continuous (a constant function is of the form $f(x) = d$; it is a "linear" function in which $c = 0$).

We now generalize this exercise to prove that every polynomial function is continuous.

Theorem 37.14. *Let $n \geq 0$ be an integer. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a polynomial function of degree at most n; that is, a function of the form*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

with $a_i \in \mathbb{R}$ for each integer i satisfying $0 \leq i \leq n$. Then f is continuous at every real number.

Proof. Fix $P(n)$ to be the open sentence

Every polynomial function of degree at most n is continuous.

We prove the theorem by induction on the integer $n \geq 0$.

Base Case: We have proved that $P(0)$ is true in Exercise 36.2.

Inductive step: Assume that $P(k)$ is true, for some integer $k \geq 0$. We will show that $P(k+1)$ now follows.

Let f be any polynomial of degree at most $k+1$, and write

$$f(x) = a_{k+1}x^{k+1} + a_k x^k + \cdots + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_{k+1} \in \mathbb{R}$. Then both of the (functions given by the) polynomials x^k and $a_k x^k + \cdots + a_1 x + a_0$ are continuous by the inductive hypothesis, and $a_{k+1}x$ is continuous by Exercise 36.2. Hence, the product

$$a_{k+1}x^{k+1} = (a_{k+1}x)x^k$$

is continuous by Theorem 37.12, so the sum

$$f(x) = a_{k+1}x^{k+1} + \left(a_k x^k + \cdots + a_1 x + a_0\right)$$

is also continuous, again by Theorem 37.12. Hence, $P(k+1)$ is true. □

Example 37.15. The polynomial function given by the rule

$$f(x) = 3x^4 - 6x^2 + 2x - 1$$

is continuous at every real number. △

37.E Exercises

Exercise 37.1. Prove that the function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by the rule $f(x) = \sqrt{x}$ is continuous at $a = 9$. (Hint: $\sqrt{x} - 3 = (\sqrt{x} - 3) \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \frac{x-9}{\sqrt{x}+3}$.)

Exercise 37.2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function, let $a \in \mathbb{R}$, and let $L_1, L_2 \in \mathbb{R}$. Prove that if

$$\lim_{x\to a} f(x) = L_1 \quad \text{and} \quad \lim_{x\to a} f(x) = L_2,$$

then $L_1 = L_2$. (*Warning*: You may *not* merely assert that

$$L_1 = \lim_{x\to a} f(x) = L_2,$$

and conclude that $L_1 = L_2$. This is because in Definition 36.3, we defined the notation

$$\lim_{x\to a} f(x) = L$$

using the symbol "=", but we never proved that the symbol "=" used this way satisfies the transitive property. Rather, you should use the fact that L_1 and L_2 each satisfy the condition in Definition 36.3 to prove that they are equal.)

Exercise 37.3. Taking f to be the characteristic function of $[0, 1]$, as described in Example 37.3, prove the following.

(a) For each $a \in \mathbb{R} - \{0, 1\}$, the function f is continuous at a.
(b) The function f is not continuous at 0.

Exercise 37.4. Prove Theorem 37.10 in the case that both L and M are equal to 0.

Exercise 37.5. Let f be a function of the form

$$f(x) = \frac{g(x)}{h(x)},$$

where $g(x)$ and $h(x)$ are polynomials. Prove that f is continuous at all points where it is defined. (You may use Theorem 37.12.)