Math 314 Test 3, Example Problems #1

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These questions are not from previous exams. These are questions that cover many, but not necessarily all, of the concepts which could appear on the exams. If you feel there is an error with the solutions, please contact the Math Lab via mathlabupper@mathematics.byu.edu, and we will rectify the mistake.

Selected Formulas

Cross Products: $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$

Second Derivative Test

Critical points: The point (a, b) is a critical point if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or they do not exist.

Given $D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$ and that (a, b) is a critical point, then:
(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is a saddle point.
(d) If $D = 0$, we know nothing about $f(a, b)$.

Multiple Choice

1. In the integral $\int_A^B \int_C^D \int_E^F g(x, y, z)dzdxdy$, which one of the following describes the expressions we can replace E with?

   A. Cannot be a function of x, y, or z
   B. Can be a function of x, and only x
   C. Can be a function of y, and only y
   D. Can be a function of z, and only z
   E. Can be a function of x and y, but not z
   F. Can be a function of x and z, but not y
   G. Can be a function of y and z, but not x
   H. Can be a function of x, y, and z

2. Find the Jacobian of the transformation $x = u/v$, $y = v/w$, $z = w/u$.

   A. 0 B. $1/uvw$ C. $-1/uvw$ D. $2/uvw$ E. $-2/uvw$

Free Response

1. A particle moves in a velocity field $\mathbf{v}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position (2,1) at time $t = 3$, estimate its location at time $t = 3.01$.

2. Find the curl and divergence of the vector fields:

   (a)
   $$\mathbf{F}(x, y, z) = \langle \ln(2y + 3z), \ln(x + 3z), \ln(x + 2y) \rangle$$

   (b)
   $$\mathbf{F}(x, y, z) = \langle e^x \sin(y), e^y \sin(z), e^z \sin(x) \rangle$$

3. Prove the following identity, assuming that all partial derivatives are continuous, f is a scalar field, and $\mathbf{F}$ is a vector field.
   $$\text{div}(f\mathbf{F}) = f\text{div}\mathbf{F} + \mathbf{F} \cdot \nabla f$$

4. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4, x \geq 0$. If the linear density is a constant k, find the mass and center of mass of the wire.

5. Evaluate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z^4\mathbf{k}$, where S is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation.

6. Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

$$\mathbf{F}(x, y) = (xy\cos(xy) + \sin(xy))\mathbf{i} + (x^2\cos(xy))\mathbf{j}$$

7. If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid*, with parametric equations $x = 5\cos(t) - \cos(5t)$, $y = 5\sin(t) - \sin(5t)$. Use Green's Theorem to find the area enclosed by the epicycloid.

8. Find a parametric representation for the plane that passes through the point $(1, 2, -3)$ and contains the vectors $\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

9. Prove the following identity, assuming that all partial derivatives are continuous, and that $\mathbf{F}, \mathbf{G}$ are vector fields.

$$\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl}\mathbf{F} - \mathbf{F} \cdot \text{curl}\mathbf{G}$$

10. Find the flux through the surface S of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ given the following vector field.

$$\mathbf{F}(x, y, z) = (\cos(z) + xy^2)\mathbf{i} + xe^{-z}\mathbf{j} + (\sin(y) + x^2 z)\mathbf{k}$$

11. Assume that S is a surface that bounds a closed region and that $\mathbf{F}$ has continuous partial derivatives. Prove that $\int\int_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = 0$.

12. Find the volume of the solid that lies within both the cylinder $x^2+y^2 = 1$ and the sphere $x^2+y^2+z^2 = 4$.

13. Evaluate $\int\int_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = 2y\mathbf{i} + e^x \sin(z)\mathbf{j} + x3^y\mathbf{k}$, and S is the hemisphere $x^2+y^2+z^2 = 9, z \geq 0$, oriented upwards.

14. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}e^{-(x^2+y^2+z^2)}dxdydz = 2\pi$$

15. Sketch the solid whose volume is given by the integral, and then evaluate the integral.

$$\int_0^4\int_0^{2\pi}\int_r^4 r dz d\theta dr$$

16. Find the area of the surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \leq u \leq 1, 0 \leq v \leq 2$

17. Find the moments of inertia I_x, I_y, I_z, and I_0 for a cube with side length L if one vertex is located at the origin and three edges lie on the coordinate axis. (Density is constant)

18. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = x^2y\mathbf{i} + \frac{1}{3}x^3\mathbf{j} + xy\mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counter-clockwise as viewed from above.

19. Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

(a)

$$\int_C y^4 dx + 2xy^3 dy$$

where C is the ellipse $x^2 + 2y^2 = 2$

(b)

$$\int_C (x^2 + y^2)dx + (x^2 - y^2)dy$$

where C is the triangle with vertices $(0, 0)$, $(2, 1)$, and $(0, 1)$.

20. Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$, where $a > 0$, if S has constant density K.

21. Determine whether or not the vector field $\mathbf{F}(x, y, z) = y\cos(xy)\mathbf{i} + x\cos(xy)\mathbf{j} + \sin(z)\mathbf{k}$ is conservative. If so, find a function f such that $\mathbf{F} = \nabla f$.

22. Evaluate

$$\int\int\int_E xyz dV$$

where E lies between the spheres $\rho = 2$ and $\rho = 4$ and above the cone $\phi = \pi/3$.

23. Evaluate

$$\int\int\int_B (x^2 + y^2 + z^2)^2 dV$$

where B is the ball with center at the origin and radius 5.