CONVOLUTION OPERATORS AND ENTIRE FUNCTIONS WITH SIMPLE ZEROS

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Abstract. Let $G(z)$ be an entire function of order less than 2 that is real for real z with only real zeros. Then we classify certain distribution functions F such that the convolution $(G * dF)(z) = \int_{-\infty}^{\infty} G(z - is)\, dF(s)$ of G with the measure dF has only real zeros all of which are simple. This generalizes a method used by Pólya to study the Riemann zeta function.

1. Introduction

In this paper we continue the investigation [Car99] of the first author on the effect of certain operators on entire functions having all of their zeros on a line. The main result of this paper is the following theorem:

Theorem 1. *Suppose G is an entire function of order < 2 that is real on the real axis and has only real zeros. Let $\{a_i\}$ be a nonincreasing sequence of positive real numbers, let $\{X_i\}$ be the sequence of independent random variables such that X_i takes values ± 1 with equal probability, and let F_n be the distribution function of the normalized sum $Y_n = (a_1 X_1 + \cdots + a_n X_n)/s_n$ where $s_n^2 = a_1^2 + \cdots + a_n^2$. The functions F_n converge pointwise to a continuous distribution $F = \lim_{n\to\infty} F_n$. Define H by the integral*

$$H(z) = (G * dF)(z) = \int_{-\infty}^{\infty} G(z - is)\, dF(s).$$

Then H is an entire function of order < 2 that is real on the real axis. If H is not identically zero, then H has only real zeros. Furthermore, the zeros of H are simple.

This theorem is stated in [Car99] without the simplicity condition. The new result in this paper is that the zeros are also *simple*. The proof of Theorem 1 is given in §3. Several examples and comments are given in §4.

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Theorem 1 generalizes a fact used by Pólya in an attempt to understand the Riemann Hypothesis. Let $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ where, for $\Re(s) > 1$, the Riemann zeta function $\zeta(s)$ is defined by $\sum_{n=1}^{\infty} n^{-s}$. The function $\xi(s)$ extends to an entire function and satisfies the functional equation $\xi(s) = \xi(1-s)$. Riemann conjectured [Rie59] that the zeros of $\xi(s)$ lie on the line in the complex plane with real part 1/2. Among the reasons for the intense interest in the Riemann Hypothesis is that its truth would give a much better error term in the Prime Number Theorem than is currently known to hold. The validity of the Riemann Hypothesis is equivalent to

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log x)$$

where $\mathrm{Li}(x)$ is the principal value integral $\int_0^x dt/\log(t)$ and $\pi(x)$ is the number of primes up to size x. The reader is referred to the texts [Edw74], [Ivi85], [Pat88], [Tit86] for basic theory of the Riemann zeta function.

In 1926 Pólya [Pól26] approximated the first term in a rapidly converging series for the function $\xi(s)$ obtaining

$$\xi(\tfrac{1}{2}+it) \approx \xi^*(\tfrac{1}{2}+it) = 4\pi^2\big(K_{it/2+9/4}(2\pi) + K_{it/2-9/4}(2\pi)\big)$$

where $K_z(u) = \int_0^\infty \exp(-u\cosh(w))\cosh(zw)dw$ is the K-bessel function. He showed that all of the zeros of $K_{it/2}(2\pi)$ are real. By applying Proposition 2 (stated below) with $a = 9/2$ and $b = 0$ he showed that the 'fake' zeta function $\xi^*(s)$ has zeros only on the line $\Re(s) = 1/2$. Additionally, let $N(T)$ be the number of zeros of the $\xi(s)$ in the strip $0 < \Re(s) < 1$ and $0 < \Im(s) < T$ and let $N^*(T)$ be the number of zeros of $\xi^*(s)$ in the same region. Then $N(T) \sim N^*(T)$. Furthermore, the zeros of $\xi^*(s)$ are simple in agreement with empirical evidence for the zeros of $\xi(s)$. This tantalizing result has been extended in [Pól27] and [Hej90].

Theorem 1 generalizes the following observation by Pólya which was needed in his analysis of the fake zeta function $\xi^*(s)$.

Proposition 2 (Pólya [Pól26], Hilfssatz II)**.** *Let a be a positive constant, let b be real, and let $G(z)$ be an entire function of genus 0 or 1 that for real z takes real values, has at least one real zero, and has only real zeros. Then the function*

$$e^{ib}G(z+ia) + e^{-ib}G(z-ia)$$

has only real zeros.

For completeness we include Pólya's short proof.

Proof. By hypothesis G has a Weierstrass product of the form

$$G(z) = cz^q e^{\alpha z}\prod\left(1 - z/\alpha_n\right)e^{z/\alpha_n}$$

where q is a nonnegative integer, c and α are real, and the α_n are the nonzero real zeros of G. Suppose $z = x + iy$ is a zero of

$$e^{ib}G(z + ia) + e^{-ib}G(z - ia).$$

Then

$$|G(z + ia)| = |G(z - ia)|$$

and

$$1 = \left|\frac{G(z - ia)}{G(z + ia)}\right|^2 = \left(\frac{x^2 + (y - a)^2}{x^2 + (y + a)^2}\right)^q \prod \frac{(x - \alpha_n)^2 + (y - a)^2}{(x - \alpha_n)^2 + (y + a)^2}.$$

If $y > 0$ then the right hand side of the last expression is < 1. If $y < 0$ then the right hand side of the last expression is > 1. Both of these cases are impossible. Hence, $y = 0$ and $e^{ib}G(z+ia)+e^{-ib}G(z-ia)$ has only real zeros. □

2. Important Notation and Preliminaries

The Laguerre-Pólya class $\mathcal{LP}$ of functions consists of the entire functions with a Weierstrass factorization of the form

$$az^q e^{\alpha z - \beta z^2} \prod (1 - z/\alpha_n) e^{z/\alpha_n}$$

where a, α, β are real, $\beta \geq 0$, q is a nonnegative integer, and the α_n are nonzero real numbers such that $\sum_{n=1}^{\infty} \alpha_n^{-2} < \infty$. We will be most interested in the subset $\mathcal{LP}^*$ of the Laguerre-Pólya class consisting of all elements of $\mathcal{LP}$ of order < 2. For functions in $\mathcal{LP}^*$, β is necessarily zero.

We will consider the following types of random variables and their distribution functions: Let $\{a_i\}$ be a nonincreasing sequence of positive real numbers. Let $\{X_i\}$ be a sequence of independent random variables such that X_i takes values ± 1 with equal probability. Let Y_n be the sum

$$Y_n = \frac{a_1 X_1 + \cdots + a_n X_n}{s_n}$$

where $s_n^2 = a_1^2 + \cdots + a_n^2$. F_n will denote the distribution function of Y_n. Iteration of the formula in Proposition 2 with the values $a_1/s_n, \ldots, a_n/s_n$ in place of the constant a and with $b = 0$ results in an expression of the type

$$H_n(z) = 2^{-n} \sum G\big(z - i(\pm a_1 \pm a_2 \cdots \pm a_n)/s_n\big),$$

where the sum is over all possible sign combinations. This may be written as the Riemann-Stieltjes integral

$$H_n(z) = (G * dF_n)(s) = \int_{-\infty}^{\infty} G(z - is) dF_n(s)$$

using the measure determined by F_n. This is a convolution of G with the measure $dF_n(s)$ along the imaginary axis. The sequence F_n converges to a limiting distribution as described in the following lemma:

Lemma 1 ([Car99])**.** *The sequence F_n converges pointwise to a continuous distribution function F. If the sequence s_n is unbounded, F is a normal distribution. If the sequence s_n is bounded, F is not a normal distribution.*

Although we have normalized the random variables to have variance $\sigma^2 = 1$, by simply rescaling, we may suppose the variance σ^2 takes any positive value. If F is a normal distribution, we will sometimes write $F = N_{\sigma^2}$ to make the dependence on the variance explicit. Then

$$F(s) = N_{\sigma^2}(s) = \int_{-\infty}^{x} dN_{\sigma^2}(x) = \int_{-\infty}^{s} \frac{e^{-x^2/(2\sigma^2)}}{\sqrt{2\pi}\,\sigma} dx.$$

3. Proof of Theorem 1

In this section G will always be a function in $\mathcal{LP}^*$ and F will always be the limit $\lim_{n\to\infty} F_n$ described in §2. There are two main parts of the proof of Theorem 1. First, it must be shown that the zeros of $H(z) = \int_{-\infty}^{\infty} G(z-is)dF(s)$ are real. We state this as

Proposition 3. *For $G \in \mathcal{LP}^*$, the function*

$$H(z) = (G * dF)(z) = \int_{-\infty}^{\infty} G(z-is)\, dF(s)$$

is also in $\mathcal{LP}^*$.

Proof. Full details of this argument are carried out in [Car99]. The basic strategy is to observe that Proposition 2 may be applied repeatedly any finite number of times. That is, if $H_n(z) = \int_{-\infty}^{\infty} G(z-is)dF_n(s)$, then $H_n(z) \in \mathcal{LP}^*$. Next it is shown that the limit $\lim_{n\to\infty} H_n(z) = H(z)$ is uniform on compact sets. By Hurwitz's Theorem the zeros of $H(z)$ are limit points of the zeros of $H_n(z)$ which are real. □

We now give the second part of the proof of Theorem 1 which will show that the zeros of $H(z) = \int_{-\infty}^{\infty} G(z-is)dF(s)$ are simple. The proof will be broken up into a sequence of lemmas. Lemmas 2 through 4 deal with the case $\sum_{i=1}^{\infty} a_i^2 < \infty$. Lemmas 5 through 9 deal with the case $\sum_{i=1}^{\infty} a_i^2 = \infty$.

Lemma 2. *If $G \in \mathcal{LP}^*$, then $G' \in \mathcal{LP}^*$.*

Proof. G' has order < 2 because differentiation does not increase the order. Since $G(z)$ is real for real z, so is $G'(z)$. By Proposition 2, the difference quotient

$$\frac{G(z+ia) - G(z-ia)}{2ia}$$

has only real zeros. The limit as $a \to 0$ is uniform for z in compact sets. So, by Hurwitz's Theorem, G' has only real zeros. Thus, $G' \in \mathcal{LP}^*$. $\square$

Lemma 3. *Let $G \in \mathcal{LP}^*$ and let $a > 0$. If*

$$G(z + ia) + G(z - ia)$$

is not identically zero, then its zeros are simple.

Proof. Suppose, by way of contradiction, that $G(z + ia) + G(z - ia)$ has a non-simple zero which, we may suppose, occurs at $z = 0$. After multiplying G by an appropriate constant if necessary, we may write

$$G(z + ia) + G(z - ia) = z^n(1 + f(z)) \tag{1}$$

where $n \geq 2$, f is entire, and $f(0) = 0$. Assume z is bounded to be inside of a disk for radius $R > 0$ about the origin. We will perturb a by real ϵ where $0 < |\epsilon| < a$.

$$\begin{aligned}
&G(z + i(a + \epsilon)) + G(z - i(a + \epsilon)) \\
&= G(z + ia) + G(z - ia) + \big(G'(z + ia) - G'(z - ia)\big)(i\epsilon) + O(\epsilon^2) \\
&= z^n(1 + f(z)) + \big(G'(ia) - G'(-ia) + O(z)\big)(i\epsilon) + O(\epsilon^2) \\
&= z^n + \big(G'(ia) - G'(-ia)\big)(i\epsilon) + O(z^{n+1}) + O(z\epsilon) + O(\epsilon^2).
\end{aligned}$$

Evaluating the derivative of equation (1) at $z = 0$ gives $G'(ia) = -G'(-ia)$. By Lemma 2, $G'(ia)$ is *not* zero. Then

$$\begin{aligned}
&G(z + i(a + \epsilon)) + G(z - i(a + \epsilon)) \\
&= z^n + 2iG'(ia)\epsilon + O(z^{n+1}) + O(z\epsilon) + O(\epsilon^2).
\end{aligned}$$

The last expression will have roots near the nth roots of $-2iG'(ia)\epsilon$ provided that ϵ is small enough. We will prove this with an application of Rouché's Theorem. Write $A = -2iG'(ia)$ and restrict z so that

$$|z| < 2|A\epsilon|^{1/n}.$$

Then

$$G(z + i(a + \epsilon)) + G(z - i(a + \epsilon)) = z^n - A\epsilon + O(\epsilon^{1+1/n}).$$

Set $g(z)$ equal to the last expression and let $h(z) = z^n - A\epsilon$. Let $\alpha = (A\epsilon)^{1/n}$ denote one of the non-real n roots of $h(z)$. Such a choice always exists if $n \geq 3$. If $n = 2$, we can choose the sign of ϵ to guarantee that $h(z)$ has non-real roots. Let β be a real number such that $1 < \beta < 1 + 1/n$, and let γ denote the path circling α parametrized by $z = (A\epsilon + e^{i\theta}|\epsilon|^\beta)^{1/n}$ for $0 \leq \theta \leq 2\pi$ and for the appropriate nth root. When ϵ is sufficiently small this path does not include any other roots of $h(z)$ or any points on the real line. For $z \in \gamma$,

$$|g(z) - h(z)| = O(\epsilon^{1+1/n}) \quad \text{and} \quad |h(z)| = |\epsilon|^{\beta}.$$

Then by letting ϵ be sufficiently small we obtain

$$|g(z) - h(z)| < |h(z)|$$

for all $z \in \gamma$. The path γ encloses a single non-real root of $h(z)$. By Rouché's Theorem,

$$g(z) = G(z + i(a + \epsilon)) + G(z - i(a + \epsilon))$$

also has a non-real root inside the path γ. However, Proposition 2 says that the zeros of $G(z + i(a + \epsilon)) + G(z - i(a + \epsilon))$ are real. This contradiction shows that all the zeros of $G(z + ia) + G(z - ia)$ are simple. □

Now we can prove one case of Theorem 1 in the next lemma. We continue to use the notation described in §2.

Lemma 4. *Suppose $\sum_{i=1}^{\infty} a_i^2 < \infty$ and $G \in \mathcal{LP}^*$. If*

$$H(z) = \int_{-\infty}^{\infty} G(z - is)dF(s)$$

is not identically zero, then the zeros of $H(z)$ are real and simple.

Proof. F is the distribution function for the random variable

$$Y = \frac{1}{\sigma}(a_1X_1 + a_2X_2 + a_3X_3 + \cdots)$$

where $\sigma^2 = \sum_{i=1}^{\infty} a_i^2$. Let F' denote the distribution function for the random variable

$$Y' = \frac{1}{\sigma}(a_2X_2 + a_3X_3 + \cdots).$$

By Proposition 3, the function

$$K(z) = \int_{-\infty}^{\infty} G(z - is)dF'(s)$$

is in $\mathcal{LP}^*$. Then

$$H(z) = \int_{-\infty}^{\infty} G(z - is)dF(s) = \frac{1}{2}\big(K(z - i(a_1/\sigma)) + K(z + i(a_1/\sigma))\big).$$

If H is not identically zero, Proposition 2 and Lemma 3 show that $H(z)$ has simple, real zeros. □

We now turn our attention to the proof of Theorem 1 when $s_n^2 = \sum_{i=1}^n a_i^2$ is unbounded. In this case, as mentioned in Lemma 1, $F = N_1$ is the normal distribution with variance 1. We will prove this case in a way similar to the proof of Lemma 3 by considering convolution of G with the distribution $N_{1-\epsilon^2}$ for small ϵ. The expression $\int_{-\infty}^{\infty} G(z-is)dN_{1-\epsilon^2}(s)$ must have real zeros by Proposition 3. However, assuming that $\int_{-\infty}^{\infty} G(z-is)dN_1(s)$ has a multiple zero will cause $\int_{-\infty}^{\infty} G(z-is)dN_{1-\epsilon^2}(s)$ to have a nonreal zero which is a contradiction. Thus, the zeros of $\int_{-\infty}^{\infty} G(z-is)dN_1(s)$ are simple. The remaining lemmas will make this argument precise. Lemmas 5 through 8 highlight several basic, but useful, observations. Lemma 9 finishes the proof of Theorem 1.

Lemma 5. *For $G \in \mathcal{LP}^*$, $(G * N_{\alpha^2}) * N_{\beta^2} = G * N_{\alpha^2+\beta^2}$.*

Proof. This results from the observation that the sum of a normal random variable with variance α^2 with another normal random variable of variance β^2 is a normal random variable of variance $\alpha^2 + \beta^2$. □

The next lemma shows that the effect of convolution with a normal distribution can be undone.

Lemma 6. *For $G \in \mathcal{LP}^*$, if*

$$H(z) = \int_{-\infty}^{\infty} G(z-is)dN_{\sigma^2}(s),$$

then

$$G(z) = \int_{-\infty}^{\infty} H(z-s)dN_{\sigma^2}(s).$$

Notice that the second integral is a convolution along the real axis instead of the imaginary axis.

Proof. This is a standard exercise in integration using the mean value property for analytic functions. □

An immediate corollary of the previous two lemmas is the following:

Lemma 7. *Assume $G \in \mathcal{LP}^*$ and let $0 < \epsilon < \sigma$. Let*

$$H(z) = \int_{-\infty}^{\infty} G(z-is)dN_{\sigma^2}(s).$$

Then

$$\int_{-\infty}^{\infty} G(z-is)dN_{\sigma^2-\epsilon^2}(s) = \int_{-\infty}^{\infty} H(z-s)dN_{\epsilon^2}(s).$$

Proof. By Lemma 5,

$$H = G * dN_{\sigma^2} = G * (dN_{\sigma^2-\epsilon^2} * dN_{\epsilon^2}) = (G * dN_{\sigma^2-\epsilon^2}) * dN_{\epsilon^2}.$$

Then apply Lemma 6. □

We will need to understand the convolution of the function z^n against the normal distribution. Define the polynomial $p_{n,\sigma}(z)$ for $\sigma > 0$ by

$$p_{n,\sigma}(z) = \int_{-\infty}^{\infty} (z-s)^n dN_{\sigma^2}(s) = \int_{-\infty}^{\infty} (z-s)^n \frac{e^{-s^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma} ds.$$

Also we will write

$$P_n(z) = p_{n,1}(z).$$

These polynomials are, in fact, expressible in terms of the Hermite polynomial of degree n. The nth Hermite polynomial $h_n(z)$ may be defined (see [Sze75]) as

$$h_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} \left(e^{-z^2}\right).$$

Then

$$p_{n,\sigma}(z) = \left(\tfrac{\sigma}{i\sqrt{2}}\right)^n h_n\left(\tfrac{i}{\sigma\sqrt{2}}\, z\right).$$

From the classical theory of orthogonal polynomials it is well known that the roots of Hermite polynomials are real and simple; therefore, the roots of $p_{n,\sigma}(z)$ are imaginary and simple. Our proof does not rely on these known facts about Hermite polynomials. Instead, we independently derive any needed facts about the zeros of these polynomials.

Lemma 8.

(a) *If n is even, $p_{n,\sigma}(z) = p_{n,\sigma}(-z)$. If n is odd, $p_{n,\sigma}(z) = -p_{n,\sigma}(-z)$.*
(b) *All roots of $p_{n,\sigma}(z)$ are of the form αi for $\alpha \in \mathbb{R}$.*
(c) *For $n \geq 2$, $p_{n,\sigma}(z)$ has a root of the form αi where $\alpha > 0$.*
(d) *$p_{n,\sigma}(z) = \sigma^n P_n(z/\sigma)$.*

Proof. Parts (a) and (d) are immediate from the definitions of $p_{n,\sigma}(z)$ and $P_n(z)$. Proposition 3 is stated for convolutions along the imaginary axis. However, after changing variables, the proposition applies to functions whose zeros are on the imaginary line and the convolution is performed along the real axis. Therefore, $p_{n,\sigma}(z)$ has only imaginary zeros. This proves (b). For part (c) we notice that $p_{n,\sigma}(z)$ is an nth degree polynomial that is not a constant times z^n. Thus it has a root different different from zero. By (b) this root is of the form αi for some real $\alpha \neq 0$. By (a) both αi and $-\alpha i$ are roots. So, we may suppose $\alpha > 0$. □

We are finally ready to complete the proof of Theorem 1.

Lemma 9. *Suppose* $\sum_{i=1}^{\infty} a_i^2 = \infty$ *and* $G \in \mathcal{LP}^*$. *If*

$$H(z) = \int_{-\infty}^{\infty} G(z - is)dF(s)$$

is not identically zero, then the zeros of $H(z)$ *are real and simple.*

Proof. Suppose, by way of contradiction, that

$$H(z) = (G * dN_1)(z) = \int_{-\infty}^{\infty} G(z - is)dN_1(s)$$

has a real zero that is not simple. Without loss of generality we may suppose that this zero occurs at $z = 0$ and that

$$H(z) = z^n + f(z)$$

where $n \geq 2$ and f vanishes to order $n+1$ at $z = 0$. We now perturb the variance by ϵ. For $0 < \epsilon < 1$, let

$$K(z) = (G * N_{1-\epsilon^2})(z).$$

By Proposition 3 the zeros of $K(z)$ are real. Using Lemma 7 we may also calculate $K(z)$ as

$$\begin{aligned} K(z) &= \int_{-\infty}^{\infty} H(z - s)dN_{\epsilon^2}(s) \\ &= \epsilon^n P_n(z/\epsilon) + \int_{-\infty}^{\infty} f(z - s)dN_{\epsilon^2}(s) \end{aligned}$$

We will apply Rouché's Theorem to the pair of functions $K(z)$ and $\epsilon^n P_n(z/\epsilon)$. Let αi denote the root of $P_n(z)$ with largest positive α. Such a root exists by Lemma 8. Choose γ to be a circular path centered at αi with small enough radius so that γ does not contain any other root of $P_n(z)$ or any points on the real axis. Then there exist positive κ_1 and κ_2 such that

$$\kappa_1 < |P_n(z)| < \kappa_2$$

for all $z \in \gamma$. Let $\epsilon\gamma$ denote the path obtained from γ by multiplying the points of γ by ϵ. The path $\epsilon\gamma$ encircles the root $\epsilon\alpha i$ of $P_n(z/\epsilon)$ and contains no other roots of $P_n(z/\epsilon)$ or any points on the real axis. Then

$$\epsilon^n \kappa_1 < |\epsilon^n P_n(z/\epsilon)| < \epsilon^n \kappa_2$$

for all $z \in \epsilon\gamma$. On the other hand, for $z \in \epsilon\gamma$,

$$|K(z) - \epsilon^n P(z/\epsilon)| = \left| \int_{-\infty}^{\infty} f(z - s)dN_{\epsilon^2}(s) \right| = O(\epsilon^{n+1})$$

since $f(z) = O(z^{n+1})$. Therefore, by taking ϵ to be sufficiently small the inequality

$$|K(z) - \epsilon^n P(z/\epsilon)| < |\epsilon^n P(z/\epsilon)|$$

is true on the path $\epsilon\gamma$. Rouché's Theorem implies that $K(z)$ has a nonreal root contained inside the path $\epsilon\gamma$. This contradicts the fact that $K(z)$ has only real roots. Therefore, we conclude that the roots of $H(z) = \int_{-\infty}^{\infty} G(z-is)dN_1(s)$ are simple and real. $\square$

This completes the proof of Theorem 1.

4. Examples and Questions

1. We may normalize the random variables Y_n in Theorem 1 to have variance σ^2 instead of 1. Let F_{σ^2} denote the resulting limit distribution and write $F = F_1$. Suppose $G \in \mathcal{LP}^*$. By letting σ vary from 0 to 1 we deform the function G which has real zeros to the function

$$H(z) = \int_{-\infty}^{\infty} G(z-is)dF(s)$$

which has simple, real zeros. This shows that the convolution causes zeros to *repel*.

2. Can the function

$$\Xi(t) = \xi(1/2+it)$$

where

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

be realized as a convolution

$$\Xi(t) = (G * dF)(t)$$

for some appropriate choice of G as in Theorem 1? This would prove the Riemann Hypothesis. Pólya's "fake" zeta function $\xi^*(\frac{1}{2}+it)$ is a convolution of this type which is why we have generalized his technique of Proposition 2 in this paper. Pólya, however, did not claim that his method was likely to lead to a proof of the Riemann Hypothesis.

3. Let $G(z) = z^n$. Proposition 2 implies that the zeros of $\frac{1}{2}\big(G(z+ia) + G(z-ia)\big)$ are real, and Lemma 3 implies that the zeros are simple. This polynomial can be explicitly factored as

$$\frac{1}{2}\big(G(z+ia)+G(z-ia)\big) = \prod_{k=0}^{n-1}\left(z - a\cot\left(\frac{(2k+1)\pi}{2n}\right)\right).$$

4. Let $G(z)$ be either $\sin(z)$ or $\cos(z)$. Then

$$\frac{1}{2}\big(G(z+ia)+G(z-ia)\big) = \cosh(a)\,G(z).$$

For a distribution F, as in Theorem 1,

$$(G * dF)(z) = \lambda\, G(z)$$

where $\lambda = \int_{-\infty}^{\infty} \cosh(s) dF(s)$. Thus, the sine and cosine functions are eigenfunctions of convolution with the measure dF. This is not surprising in light of the first example which shows that convolution causes zeros to repel. Since the zeros of the sine and cosine are evenly spaced the repulsion of zeros does not alter the location of the zeros.

5. Let

$$Y_n = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{2n}}$$

were $X_i = \pm 1$. The distribution function for Y_n converges to the normal distribution with variance $1/2$. That is,

$$F(s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{s} e^{-u^2} du.$$

Let $G(z) = z^n$. Then

$$\begin{aligned} (G * dF)(z) &= \int_{-\infty}^{\infty} (z - is)^n dF(s) \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (z - is)^n e^{-s^2} ds \\ &= 2^{-n} h_n(z) \end{aligned}$$

where $h_n(z)$ is the nth degree Hermite polynomial defined by

$$h_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} \left(e^{-z^2} \right).$$

Therefore, Theorem 1 gives a new proof that the zeros of Hermite polynomials are simple and real. We do admit, however, that the classical proof involving inner products is simpler.

6. The hypotheses in Proposition 2 and Theorem 1 appear to be necessary. Relaxing them produces non-examples with non-real zeros.

(a) For arbitrary symmetric distribution functions F, Theorem 1 does not necessarily hold. The function $G(z) = z^4$ has only real zeros, but

$$H(z) = \tfrac{1}{19}\big(G(z+i) + 17G(z) + G(z-i)\big)$$

has no real zeros.

(b) The function $G(z) = ze^{z^2}$ has genus 2 which is not permitted in either Proposition 2 or Theorem 1. Then

$$H(z) = \frac{1}{2}\big(G(z+i) + G(z-i)\big)$$

does not have only real zeros. There is a zero approximately at $z = 0.957505i$.

(c) By Lemma 2, if $G \in \mathcal{LP}^*$, then the derivative G' is also in $\mathcal{LP}^*$. The function $G(z) = ze^{z^2}$ in part (b) does have only real zeros but its genus it too large. $G'(z) = (1 + 2z^2)e^{z^2}$ has non-real zeros.

7. In Theorem 1 let $a_i = 2^{-i}$. The distribution function of the random variable $Y = \sum_{i=1}^{\infty} 2^{-i} X_i$ is

$$F(x) = \begin{cases} \frac{1}{2} + \frac{x}{2} & \text{if } -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

For G as in Theorem 1,

$$H(z) = (G * dF)(z) = \frac{1}{2} \int_{-1}^{1} G(z - is) ds$$

has only simple, real zeros.

8. An interesting question to consider is how the distribution of the spacing of the zeros of G compares with the distribution of the spacing of the zeros of $G * dF$.

Experimental data obtained by Andrew Odlyzko (see for example [Odl87] and [Odl00]) has supported the Montgomery pair-correlation conjecture [Mon73] as well as the idea that the the zeros of the zeta function have a distribution corresponding to the distribution of eigenvalues in the Gaussian Unitary Ensemble. Examples of recent work relating to a spectral interpretation of the zeros of the zeta function are [KS99], [KS], and [Rub98].

Perhaps, one could hypothesize that $G * dF$ satisfies the type of distribution of zeros expected to hold for the Riemann zeta function and then derive consequences for G.

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