A RIEMANN HYPOTHESIS CONDITION FOR METAPLECTIC EISENSTEIN SERIES

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ABSTRACT. Let $\mathbb{A}$ be the adeles of the function field $\mathbb{F}_q(T)$. If $E(g,s)$ is the Eisenstein series associated with the spherical vector in the principal series representation of the double cover of $GL_2(\mathbb{A})$, then the Fourier coefficients (Whittaker functions) of $E(g,s)$ have an Euler product and satisfy a 'Riemann Hypothesis' condition. Namely, the zeros occur only on the line $\Re(s) = 0$. These Whittaker functions are essentially quadratic L-functions associated with hyperelliptic curves.

This result adds to the list of known examples for which the Fourier coefficients of Eisenstein series coming from the spherical vector in the principal series representation satisfy a Riemann Hypothesis condition.

CONTENTS

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1. Introduction

A major theme of modern number theory is the study of L-functions or zeta functions attached to an arithmetic object such as a field or variety. These functions are usually defined as a Dirichlet series of the shape

$$L(s)=\sum_{n=1}^{\infty} a_n n^{-s}$$

which converges in a right half plane. Then with suitable normalization and choice of meromorphic function $\xi(s)$ one tries to prove the following theorem:

Desired Theorem.

1. *$L(s)$ has meromorphic continuation to all $s \in \mathbb{C}$.*
2. *$L(s)$ satisfies the functional equation $\xi(s)L(s)=\xi(1-s)L(1-s)$.*
3. *$L(s)$ has an Euler product $\prod_v L_v(s)$ where the $L_v(s)$ are local factors.*
4. *The nontrivial zeros of $L(s)$ are on the line $\Re(s)=1/2$.*

The Desired Theorem is obviously patterned after the archetypical Riemann zeta function given by $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$ for $\Re(s)>1$. The meromorphic continuation, functional equation, and Euler product of $\zeta(s)$ are well known. In this case the fourth condition of the Desired Theorem is the celebrated Riemann Hypothesis. The proof that the zeta function has its nontrivial zeros on the line $\Re(s)=1/2$ has so far eluded proof. We will refer to the condition of having zeros on the line $\Re(s)=1/2$ (or sometimes $\Re(s)=0$) as a *Riemann Hypothesis condition*.

In this paper we will explain how the Fourier coefficients of certain Eisenstein series satisfy some or all of the conditions of the Desired Theorem. Because the general theory of Eisenstein series already guarantees that the Fourier coefficients have meromorphic continuation and functional equations (see [16] and [13]) we are primarily concerned with the existence of the Euler product and Riemann Hypothesis condition.

The remainder of this introductory section will motivate the theme by briefly mentioning three known examples of Eisenstein series whose Fourier coefficients satisfy the Riemann Hypothesis condition.

In Section 2 we will state the Main Theorem of this paper which will provide a fourth example of this interesting phenomenon. That is, if $\mathbb{A}$ is the adeles of the function field $\mathbb{F}_q(T)$, then the Fourier coefficients of a certain Eisenstein series on the two fold cover of $GL_2(\mathbb{A})$ is essentially a quadratic L-function which satisfies all of the conditions of the Desired Theorem.

Finally, in Section 3 we will give the proof of the Main Theorem.

We note that other authors have worked with metaplectic Eisenstein series in the function field case. Hoffstein in [8] and [9] and Gupta in [5] have calculated Fourier coefficients of Eisenstein series working with the completion of the function field at the infinite place rather than adelically as in this paper. In [8] and [5] no mention is made about the location of the zeros of these Fourier coefficients or about the possibility of finding an Euler product. In [9] page 83 Hoffstein makes the 'curious observation' that the first Fourier coefficient does have its zeros on the line $\Re(s) = 1/2$. Although some of the Fourier coefficients calculated in [8], [9], and [5] will satisfy the Riemann Hypothesis condition and have an Euler product generally they do not.

In this paper we do the Fourier analysis with an abelian subgroup of the metaplectic group which is slightly larger than the abelian subgroup used in the corresponding non-adelic case treated by Hoffstein and Gupta. Consequently, we are able to 'repair' the Fourier coefficients so that every coefficient both has an Euler product and satisfies the Riemann Hypothesis condition.

1.1. **Example 1 – Classical $SL_2(\mathbb{Z})$ Eisenstein Series.** The classical nonholomorphic $SL_2(\mathbb{Z})$ Eisenstein series is given by

$$E(z,s) = \frac{1}{2}\pi^{-s}\Gamma(s)\sum_{c,d}\frac{y^s}{|cz+d|^{2s}}$$

where $z = x + iy \in \mathbb{C}$ has positive imaginary part and the summation is over all pairs of integers c and d not simultaneously zero. A complete discussion of this Eisenstein series can be found in Bump's excellent book *Automorphic Forms and Representations* [1]. The series converges absolutely when $\Re(s) > 1$. It can be shown that there is meromorphic continuation to all $s \in \mathbb{C}$ with the exception of simple poles at $s = 0$ and $s = 1$. There is a functional equation $E(z,s) = E(z,1-s)$. Of paramount importance is the automorphic property of $E(z,s)$ with respect to the group $SL_2(\mathbb{Z})$. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, then

$$E\left(\frac{az+b}{cz+d}, s\right) = E(z,s).$$

Because $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ it becomes apparent that $E(z+1,s) = E(z,s)$ so that there is a Fourier series expansion of $E(z,s)$. The nth Fourier coefficient is

$$\begin{aligned} a_n(y,s) &= \int_0^1 E(x+iy,s)e^{-2\pi inx}\,dx \\ &= 2\sqrt{y}\,|n|^{s-1/2}\sigma_{1-2s}(|n|)\,K_{s-1/2}(2\pi|n|y) \end{aligned}$$

where $\sigma_\nu(n)$ is the divisor function

$$\sigma_\nu(n) = \sum_{d|n} d^\nu$$

and $K_\nu(y)$ is the K-Bessel function

$$K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})}t^\nu\frac{dt}{t}.$$

The Fourier coefficients $a_n(y, s)$ are easily seen to satisfy the functional equation $a_y(n, s) = a_y(n, 1-s)$. There is an Euler product in the sense that the divisor function factors over primes and the Bessel function may be regarded as the factor coming from the infinite prime. The divisor function $\sigma_\nu(n)$ has zeros only if $\Re(\nu) = 0$, and Pólya has shown in [17] that $K_\nu(y)$ has its zeros on the line $\Re(\nu) = 0$. Hence, if $n \neq 0$, then $a_n(y, s)$ satisfies all the conditions of the Desired Theorem.

1.2. **Example 2 – The Double Cover of $GL_2(\mathbb{C})$.** The Eisenstein series on a double cover of $GL_2(\mathbb{C})$ associated with the imaginary quadratic extensions of $\mathbb{Q}(i)$ and their Fourier coefficients were studied by Kubota in [11] and [12]. Hoffstein has summarized Kubota's results in [9]. Kubota found that certain Fourier coefficients are expressible in terms of some gamma factors, a Riemann zeta function factor, a quadratic L-function, and a K-Bessel function. If the generalized Riemann Hypothesis holds then some of these Fourier coefficients will have their zeros on the half line $\Re(s) = 1/2$.

1.3. **Some Notation.** Before giving the third example we will define some important notation that will be used throughout the rest of the paper.

The finite field of q elements is $\mathbb{F}_q$, $R = \mathbb{F}_q[T]$ is the polynomial ring in the indeterminate T with coefficients in $\mathbb{F}_q$, and $F = \mathbb{F}_q(T)$ is the field of fractions of R. F is a global function field with adele ring $\mathbb{A}$. A place of F and its corresponding valuation will both be denoted by v. The completion of F at v is F_v and its ring of integers is $\mathfrak{O}_v$. If v is a finite place we may choose a local uniformizing parameter π_v to correspond to a monic irreducible polynomial in $\mathbb{F}_q[T]$ regarded as an element of $\mathbb{A}$ in the standard way. If v is the infinite place we will sometimes write $v = \infty$ and we may choose the uniformizing parameter π_∞ in F_∞ to be $1/T$. The group $K_v = GL_2(\mathfrak{O}_v)$ is a maximal compact subgroup of $G_v = GL_2(F_v)$, and $K = \prod_v K_v$ is a maximal compact subgroup of $G = GL_2(\mathbb{A})$. B will be the group of upper triangular matrices in G. G_F is $GL_2(F)$ and B_F is the upper triangular subgroup of G_F. The idele with π_∞ at the infinite place and ones elsewhere is

$$\tau = \tau_\infty = (\pi_\infty, 1, 1, \dots),$$

and for finite v we define τ_v as

$$\tau_v = (\pi_v, \pi_v, \pi_v, \dots)\tau^{\deg(\pi_v)}$$

where $\deg(\pi_v)$ is the degree of π_v as a polynomial in $\mathbb{F}_q[T]$. It will be convenient to say that $\deg(\pi_\infty) = 1$. We will let $q_v = q^{\deg(\pi_v)}$, $x = q^{-2s}$, and $x_v = q_v^{-2s}$ where s is complex.

Because we will be doing Fourier analysis on adele groups we will need an additive unitary character ψ on $\mathbb{A}$ which is trivial on F. By restricting ψ we obtain characters ψ_v on the local fields F_v so that $\psi = \prod_v \psi_v$. It is possible to choose ψ in such a way that for $v < \infty$ the conductor of ψ_v is

$\mathfrak{O}_v$ while for $v = \infty$ the conductor of ψ_∞ is $T^{-2}\mathfrak{O}_\infty$. In particular, for finite v, ψ_v will be the composition

$$F_v \to F_v/\mathfrak{O}_v \xrightarrow{\sim} R[1/\pi_v]/R \to F/R \to F_\infty/R \xrightarrow{\lambda} \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}^\times .$$

The first arrow is the quotient map. The second, third, and fourth arrows are canonical injections. If $x \in F_\infty/R$ is represented as $\sum_{i=1}^\infty a_i T^{-i}$ where $a_i \in \mathbb{F}_q$ then $\lambda(x) = \mathrm{Trace}_{\mathbb{F}_p}^{\mathbb{F}_q}(a_1)$ where p is the characteristic of $\mathbb{F}_q$. The last arrow is the exponential map $a \mapsto e^{2\pi i a/p}$. At the infinite place ψ_∞ is the composition

$$F_\infty \to F_\infty/R \xrightarrow{-\lambda} \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}^\times .$$

This additive character will be used throughout the rest of the paper. If ψ' is any other unitary character on $\mathbb{A}/F$, then $\psi'(x) = \psi(\alpha x)$ for some $\alpha \in F^\times$.

The additive measure on F_v will be normalized to be self dual with respect to the character ψ_v mentioned in the preceding paragraph. A simple calculation using the characteristic function of $\mathfrak{O}_v$ shows that the correct normalization for $v < \infty$ is $\mathrm{vol}(\mathfrak{O}_v) = 1$ while for $v = \infty$ it is $\mathrm{vol}(\mathfrak{O}_\infty) = q$.

If v is a finite place and $x \in F_v^\times$, let $g_v(x)$ denote the Gauss sum

$$g_v(x) = \sum_{e \in \mathbb{F}_{q_v}^\times} (e, x)_v \psi_v(e\pi_v^{-1}),$$

and if v is the infinite place let

$$g_\infty(x) = \sum_{e \in \mathbb{F}_q^\times} (e, x)_\infty \psi_\infty(e\pi_\infty)$$

where for $a, b \in F_v^\times$, $(a, b)_v$ is the local Hilbert symbol.

Let $\xi(s)$ be the meromorphic function

$$\xi(s) = q^{-2s}(1 - q^{-2s})^{-1}(1 - q^{-1-2s})^{-1}.$$

$\xi(s)$ will be used as a normalizing factor for the Whittaker functions (defined later) just as the gamma function $\Gamma(s)$ is used to normalize the Riemann zeta function to obtain a simple functional equation.

Sometimes entries of matrices that are zero will be left blank, and sometimes if a is an adele but ϕ is a function on the local field F_v we will still write $\phi(a)$. This means to evaluate ϕ at the vth component of a.

1.4. **Example 3 – $GL_2(\mathbb{A})$ Eisenstein Series.** Let f_s be the right K-invariant function on $GL_2(\mathbb{A})$ such that

$$f_s(\left(\begin{smallmatrix} a & * \\ & d \end{smallmatrix}\right)k) = |a/d|^{s+1/2}$$

for all $k \in K$ and $a, d \in \mathbb{A}^\times$. Thus, if $\chi_s : B \to G$ is the character defined by

$$\chi_s\left(\begin{smallmatrix} a & * \\ & d \end{smallmatrix}\right) = |a/d|^s,$$

then f_s is the so-called *spherical vector* in the induced representation $Ind_B^G(\chi_s)$ which is the space of smooth functions f on G with the property that

$$f(bg) = (\mu^{1/2}\chi_s)(b)f(g)$$

for every $b \in B$ and $g \in G$. We have normalized the induction by the modular character μ where $\mu(\begin{smallmatrix} a & \\ & d \end{smallmatrix}) = |a/d|$. An *Eisenstein series* may be constructed by summing translates of f_s as follows:

$$E(g, s) = \sum_{\gamma \in B_F \backslash G_F} f_s(\gamma g). \tag{1}$$

Notice that because

$$E((\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix})g, s) = E(g, s)$$

whenever $x \in F$, there is a Fourier expansion of the Eisenstein series:

$$E((\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix})g, s) = \sum_{\alpha \in F} W_\alpha(g, s)\, \bar{\psi}(\alpha x) \tag{2}$$

where

$$W_\alpha(g, s) = \int_{\mathbb{A}/F} E((\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix})g, s)\, \psi(\alpha x)\, dx$$

and where $\psi : \mathbb{A}/F \to \mathbb{C}^\times$ is a nontrivial additive unitary character. We define the *Whittaker function*, $W(g, s)$, to be

$$W(g, s) = \int_{\mathbb{A}/F} E((\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix})g, s)\, \psi(x)\, dx = \int_{\mathbb{A}} f_s((\begin{smallmatrix} & 1 \\ 1 & x \end{smallmatrix})g)\, \psi(x)\, dx.$$

In terms of the Whittaker function, the Fourier coefficients (for $\alpha \neq 0$) in equation (2) are

$$W_\alpha(g, s) = W((\begin{smallmatrix} \alpha & \\ & 1 \end{smallmatrix})g, s).$$

Because $\chi_s = \prod_v \chi_{v,s}$ and $\psi = \prod_v \psi_v$ the integral for the Whittaker function factors as

$$W(g, s) = \prod_v \int_{F_v} f_{v,s}((\begin{smallmatrix} & 1 \\ 1 & x \end{smallmatrix})g)\, \psi_v(x)\, dx. \tag{3}$$

This gives an Euler product for $W(g, s)$.

An immediate consequence of the definition of $W(g, s)$ is that

$$W(g, s) = \alpha\, W((\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}), s)$$

for some idele $a \in \mathbb{A}$ and constant α which both depend on g. Thus, for the purpose of studying zeros there is nothing lost is supposing g is a diagonal matrix as in the previous equation.

For $a \in \mathbb{A}^\times$ let $r_v = v(a)$ be the valuation of a at the vth place. Let $x_v = q_v^{-2s}$ and define the local factor $\mathcal{W}_v$ as

$$\mathcal{W}_v(\mathrm{diag}(a\tau^2, 1), s) = q_v^{-r_v/2} x_v^{-r_v/2}(1 + x_v + \cdots + x_v^{r_v})$$

and let $\xi(s)$ be the meromorphic function defined in Section 1.3. By the *normalized* Whittaker function, $\mathcal{W}(g,s)$, we mean

$$\mathcal{W}(g,s) = \xi(s)\, W(g,s).$$

We then have the following proposition:

Proposition. *If $a \in \mathbb{A}^\times$ and $|a|_v \leq 1$ for all v, then the normalized Whittaker function has an Euler product*

$$\mathcal{W}(\mathrm{diag}(a\tau^2,1),s) = \prod_v \mathcal{W}_v(\mathrm{diag}(a\tau^2,1),s)$$

and all of its zeros are on the line $\Re(s) = 0$. If $|a|_v > 1$ for some v, then $\mathcal{W}(\mathrm{diag}(a\tau^2,1),s) = 0$. Thus the nontrivial Whittaker functions satisfy the Riemann Hypothesis condition.

The proposition shows that $\mathcal{W}(g,s)$ satisfies all the conditions of the Desired Theorem. The meromorphic continuation is clear because each of the $\mathcal{W}_v(g,s)$ is meromorphic and because $\mathcal{W}_v(g,s) = 1$ for all but finitely many v. The functional equation $\mathcal{W}(g,s) = \mathcal{W}(g,-s)$ is also immediate from the proposition. The proof is an easy exercise in calculating the local integrals in equation (3). Similar integral are evaluated in Bump [1] chapter 3. The τ^2 that appears in the proposition simplifies the condition imposed on a. It is a reflection of the fact that the conductor of ψ_∞ is $\pi_\infty^2\mathfrak{O}_\infty$ rather than $\mathfrak{O}_\infty$ as mentioned in Section 1.3.

2. Whittaker Functions on $\widetilde{GL_2}(\mathbb{A})$

We will now consider Whittaker functions on the double cover of $GL_2(\mathbb{A})$ where $\mathbb{A}$ is the adele ring of the function field $\mathbb{F}_q(T)$. We will assume that q is a power of an odd prime and that -1 is a square in the finite field $\mathbb{F}_q$. So that we may state the Main Theorem precisely we will recall the definition of the metaplectic group in Section 2.1. The Eisenstein series is defined in Section 2.2, and its Fourier coefficients, the Whittaker functions, are defined in Section 2.3. Then we state the Main Theorem in Section 2.4.

2.1. **The Metaplectic Group.** Let $G_v = GL_2(F_v)$. The local metaplectic group will be a two-fold cover of G_v which is a central extension of G_v by the subgroup $\{\pm I\}$ where I is the identity matrix. Using a 2-cocycle $\beta_v : G_v \times G_v \to \{\pm 1\}$ in $H^2(G_v, \{\pm 1\})$ which is not a coboundary one may construct a central extension $\tilde{G}_v$ (see Hall [6] chapter 15). As a set $\tilde{G}_v = G_v \times \{\pm 1\}$, and the multiplication is defined by

$$\langle g_1, \zeta_1\rangle\langle g_2, \zeta_2\rangle = \langle g_1 g_2, \zeta_1\zeta_2\beta_v(g_1,g_2)\rangle. \tag{4}$$

The group $\tilde{G}_v$ is called a two-fold metaplectic cover or a double cover of G_v.

Kubota [11] constructed an explicit cocycle for G_v. Kazhdan and Patterson [10] modified Kubota's cocycle giving a formula for β_v as follows:

Assume the local field F_v contains distinct second roots of unity. Because the global field is $F = \mathbb{F}_q(T)$ this means that q must be odd. Let

$$X\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{if } c = 0. \end{cases}$$

In terms of the local Hilbert symbol the cocycle is

$$\beta_v(g, g') = \left(\frac{X(gg')}{X(g)}, \frac{X(gg')}{X(g')\det(g)}\right)_v.$$

Some important properties of the (local nth power) Hilbert symbol used in this paper are:

1. $(aa', b)_v = (a, b)_v(a', b)_v$;
2. $(a, bb')_v = (a, b)_v(a, b')_v$;
3. $(a, b)_v = 1$ if and only if b is a norm from $F_v(a^{1/n})$;
4. $(a, -a)_v = (a, 1-a)_v = 1$;
5. $(a, b)_v = (b, a)_v^{-1}$;
6. If $(a, b)_v = 1$ for all $b \in F_v^\times$, then $a \in F_v^{\times n}$.

This is proposition 14.7 in Serre [19]. In our application $n = 2$.

For ideles a and b the global Hilbert symbol is the product $(a, b) = \prod_v (a, b)_v$. If $a, b \in F^\times$, then $(a, b) = 1$. Because -1 is assumed to be a square in $\mathbb{F}_q$ we also have $(a, a) = 1$ for all $a \in \mathbb{A}^\times$.

Kazhdan and Patterson [10] have considered 'twisted' metaplectic covers $\tilde{G}^{(c)}$ for $c \in \mathbb{Z}/n\mathbb{Z}$. In this paper it will suffice to use the definition described above which corresponds to the case $c = 0$ and $n = 2$.

The definition of the global metaplectic group, $\widetilde{GL}_2(\mathbb{A})$, is similar to that of the local group $\widetilde{GL}_2(F_v)$. Essentially a cocycle $\widetilde{GL}_2(\mathbb{A}) \times \widetilde{GL}_2(\mathbb{A}) \to \{\pm 1\}$ is formed by taking the product of local cocycles. However the local cocycle requires modification by a coboundary to ensure that the product will be defined. Let $K_v = GL_2(\mathfrak{O}_v)$ where $\mathfrak{O}_v$ is the ring of integers in F_v. A new local cocycle σ_v which is trivial on $K_v \times K_v$ will be defined by multiplying by an appropriate coboundary α_v as follows

$$\sigma_v(g_v, g'_v) = \beta_v(g_v, g'_v)\alpha_v(g_v g'_v)^{-1}\alpha_v(g_v)\alpha_v(g'_v)$$

where

$$\alpha_v(g_v) = \begin{cases} \kappa_v(g_v) & \text{if } g_v \in K_v \\ 1 & \text{if } g_v \notin K_v \end{cases}$$

and $k \to \langle k, \kappa_v(k)\rangle$ is the lifting of K_v to the local metaplectic group given by Kubota in [11] as

$$\kappa_v(k) = \begin{cases} (c, d/\det(k)) & \text{if } k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ and } 0 < |c|_v < 1, \\ 1 & \text{if } |c|_v = 1 \text{ or } 0. \end{cases}$$

The product of the new cocycles σ_v does converge so that an adelic cocycle is defined by

$$\sigma(g, g') = \prod_v \sigma_v(g_v, g'_v).$$

Let $\mathbf{p}$ be the projection map from $\tilde{G}$ onto G given by $\langle g, \zeta\rangle \mapsto g$, and let $\mathbf{s}$ be the section from G into $\tilde{G}$ given by $\mathbf{s}(g) = \langle g, 1\rangle$. If G' is a subgroup of G, then G' lifts to $\tilde{G}$ if there is a homomorphism $\mathbf{h} : G' \to \tilde{G}$ such that $\mathbf{p} \circ \mathbf{h}$ is the identity map on G'. As pointed out in [10], $GL_2(F)$ and N lift to $\tilde{G}$ via the preferred section $\mathbf{s}$. There is also a lifting κ of the maximal compact subgroup $K = \prod_v GL_r(\mathfrak{O}_v)$ to $\tilde{G}$. The liftings $\mathbf{s}$ and κ are 'compatible' in the sense that

$$\begin{aligned} \kappa|_{H\cap K} &= \mathbf{s}|_{H\cap K} \\ \kappa|_{W} &= \mathbf{s}|_{W} \\ \kappa|_{N\cap K} &= \mathbf{s}|_{N\cap K}. \end{aligned}$$

This is proposition 0.I.3 of [10]. Because of these liftings $GL_r(F)$, N, and K will routinely be regarded as subgroups of $\tilde{G}$.

Although we are restricting this discussion to the two fold cover of $GL(2)$ we point out that Matsumoto [15] generalized Kubota's cocyle allowing the n-fold metaplectic cover of $SL_r(F_v)$ to be constructed. Kazhdan and Patterson [10] by regarding $GL_r(F_v) \subset SL_{r+1}(F_v)$ define the n-fold metaplectic cover of $GL_r(F_v)$ and construct Eisenstein series on these groups. This construction of the metaplectic group is also mentioned in Bump and Hoffstein [2]. Bump has pointed out that the construction given in [2] does require -1 to be an nth power in $\mathbb{F}_q$.

2.2. **The Eisenstein Series.** In order to define the Eisenstein series we must first define a certain induced representation. For a subgroup Y of the ideles let $Z(Y)$ denote the scalar matrices in $GL_2(Y)$ and let $\tilde{Z}(Y)$ be the preimage of $Z(Y)$ in $\tilde{G}$ under the projection map. Let $H(Y)$ denote the diagonal matrices in $GL_2(Y)$. Via the standard section $H(F^\times)$ is a subgroup of $\tilde{G}$. As mentioned in [10] the group

$$\tilde{M}_a = H(F^\times)\tilde{Z}(\mathbb{A}^{\times 2})$$

is a maximal abelian subgroup of $\tilde{G}$, and another maximal abelian subgroup of $\tilde{G}$ is

$$\tilde{T}_a = \{\langle \mathrm{diag}(x_1, x_2), \epsilon\rangle \in \tilde{G} \mid v(x_i) \text{ is even for all } v\}$$

where $v(x_i)$ denotes the valuation of x_i at the place v. ($\tilde{T}_a$ is the same as $\tilde{H}_*$ with $c = 0$ and $n = 2$ in [10] page 59.)

We define a genuine character χ_s on $\tilde{M}_a$ and $\tilde{T}_a$ by the formula

$$\chi_s\big(\langle \left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right), \epsilon\rangle\big) = \epsilon |a/d|^s.$$

If N is the subgroup of element of $\tilde{G}$ of the form $\langle\left(\begin{smallmatrix}1&*\\&1\end{smallmatrix}\right),1\rangle$, then we extend the definition of χ_s to $\tilde{M}_a N$ and $\tilde{T}_a N$ by

$$\chi_s\big(\langle\left(\begin{smallmatrix}a&*\\&d\end{smallmatrix}\right),\epsilon\rangle\big)=\epsilon|a/d|^s.$$

By μ we will mean the character which has the effect

$$\mu\big(\langle\left(\begin{smallmatrix}a&\\&d\end{smallmatrix}\right),\epsilon\rangle\big)=|a/d|.$$

We define the induced representation $Ind_{\tilde{M}_a N}^{\tilde{G}}(\chi_s)$ to be the vector space of smooth complex valued functions f on $\tilde{G}$ such that if $m\in\tilde{M}_a N$ and $g\in\tilde{G}$, then

$$f(mg)=(\chi_s\mu^{1/2})(m)f(g).$$

The space $Ind_{\tilde{T}_a N}^{\tilde{G}}(\chi_s)$ is similarly defined. These two induced representations are related through intertwining operators. If $f'\in Ind_{\tilde{T}_a N}^{\tilde{G}}(\chi_s)$ then we obtain a corresponding $f\in Ind_{\tilde{M}_a N}^{\tilde{G}}(\chi_s)$ by the equation

$$f(g)=\sum_{h\in H(F^{\times 2})\backslash H(F^\times)}1/4\cdot f'(hg). \tag{5}$$

Equation (5) is a special case of the intertwining operator given in [10] page 109.

The most interesting vector in $Ind_{\tilde{T}_a N}^{\tilde{G}}(\chi_s)$ is the spherical vector f'_s. It is the unique right K-invariant vector normalized so that $f'_s(\langle\left(\begin{smallmatrix}1&\\&1\end{smallmatrix}\right),1\rangle)=1$, and its formula is

$$f'_s\big(\langle\left(\begin{smallmatrix}a&*\\&d\end{smallmatrix}\right),\epsilon\rangle k\big)=\chi_s\big(\langle\left(\begin{smallmatrix}a&*\\&d\end{smallmatrix}\right),\epsilon\rangle\big)=\epsilon|a/d|^{s+1/2}$$

for all $\langle\left(\begin{smallmatrix}a&*\\&d\end{smallmatrix}\right),\epsilon\rangle\in\tilde{T}_a N$ and $k\in K$. The corresponding spherical vector $f_s\in Ind_{\tilde{M}_a N}^{\tilde{G}}(\chi_s)$ is give by equation (5). The factor $1/4$ in (5) gives the normalization $f_s(\langle\left(\begin{smallmatrix}1&\\&1\end{smallmatrix}\right),1\rangle)=1$.

Definition 1. For $f\in Ind_{\tilde{M}_a N}^{\tilde{G}}(\chi_s)$ the Eisenstein series $E(g,f,s)$ is

$$E(g,f,s)=\sum_{\gamma\in B_F\backslash G_F}f(\gamma g)$$

where $G_F=GL_2(F)$ and B_F is the subgroup of upper triangular elements of G_F. In the case of the spherical vector we will write $E(g,s)=E(g,f_s,s)$.

We are primarily interested in the Eisenstein series coming from the spherical vector.

2.3. **Fourier Series and Whittaker Functions.** From definition 1 it is clear that for all $g' \in GL_2(F)$ the Eisenstein series satisfies $E(g'g, s) = E(g, s)$. As a special case, if $x \in F$ then

$$E\big(\langle\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right), 1\rangle g, s\big) = E(g, s)$$

so that there is a Fourier series with respect to the variable x and the additive character $\psi : \mathbb{A}/F \to \mathbb{C}^\times$. That is,

$$E\big(\langle\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right), 1\rangle g, s\big) = \sum_{\alpha \in F} W_\alpha(g, s)\, \bar{\psi}(\alpha x)$$

where

$$W_\alpha(g, s) = \int_{\mathbb{A}/F} E\big(\langle\left(\begin{smallmatrix} 1 & y \\ & 1 \end{smallmatrix}\right), 1\rangle g, s\big)\, \psi(\alpha y)\, dy.$$

Definition 2. The pre-constant term is the integral

$$C(g, s) = \int_{\mathbb{A}/F} E\big(\langle\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right), 1\rangle g, s\big)\, dx,$$

and the pre-Whittaker function is the integral

$$W(g, s) = \int_{\mathbb{A}/F} E\big(\langle\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right), 1\rangle g, s\big)\, \psi(x)\, dy.$$

Notice that the Fourier coefficients are really pre-Whittaker functions so that

$$E(g, s) = C(g, s) + \sum_{\substack{\alpha \in F \\ \alpha \neq 0}} W\big(\langle\left(\begin{smallmatrix} \alpha & \\ & 1 \end{smallmatrix}\right), 1\rangle g, s\big).$$

The sum defining the Eisenstein series may be combined with the integration to give

$$W(g, s) = \int_{\mathbb{A}} f_s\big(\langle\left(\begin{smallmatrix} & 1 \\ 1 & x \end{smallmatrix}\right), 1\rangle g\big)\psi(x)\, dx.$$

The pre-Whittaker function roughly corresponds to the non-metaplectic Whittaker function from Example 3 because in both cases we have integrated on the group N. However, in the metaplectic case the pre-Whittaker function does not meet the requirements of the Desired Theorem. Generally, there is no Euler product and the Riemann Hypothesis condition is not satisfied. We will need to do Fourier analysis on a larger group than N in order to achieve satisfactory results.

To this end, let $Z_1 = Z(\mathbb{A}^\times)$ and $Z_0 = Z(F^\times \mathbb{A}^{\times 2})$. Then the subgroup Z_1 of G lifts to $\tilde{G}$. A map $\gamma : Z_1 \to \{\pm 1, \pm i\}$ defining the lift $z \mapsto \langle z, \gamma(z)\rangle$ was given explicitly in Weil [20] where he used a slightly larger metaplectic group than that used here. However, the assumption that -1 is a square in $\mathbb{F}_q$ allows the map γ to take values in $\{\pm 1\}$. Notice that γ must satisfy the relation

$$\gamma(xy) = (x, y)\gamma(x)\gamma(y).$$

There is some freedom in the definition of γ and it is possible to choose γ to be trivial on $\prod_v \mathfrak{O}_v^\times$, trivial on $F^\times$, and trivial on the element τ_∞ which will be assumed in what follows. Inside the metaplectic group Z_1 commutes with N. Also Z_1/Z_0 is both compact and Hausdorff. If $z \in Z_0$, then the Eisenstein series satisfies

$$E(zg,s) = E(g,s).$$

We may now do Fourier analysis on $E(g,s)$ using unitary characters on the abelian group Z_1N. For a given character θ on Z_1/Z_0 we make the following definition:

Definition 3. The constant term is the integral

$$\mathcal{C}(g,s) = \int_{\mathbb{A}^\times/F^\times\mathbb{A}^{\times 2}} \int_{\mathbb{A}/F} E\big(\langle\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right),1\rangle\langle\left(\begin{smallmatrix}z & \\ & z\end{smallmatrix}\right),\gamma(z)\rangle g,s\big)\,dx\,dz$$

and the Whittaker function is the integral

$$\mathcal{W}(g,s) = \int_{\mathbb{A}^\times/F^\times\mathbb{A}^{\times 2}} \int_{\mathbb{A}/F} E\big(\langle\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right),1\rangle\langle\left(\begin{smallmatrix}z & \\ & z\end{smallmatrix}\right),\gamma(z)\rangle g,s\big)\,\psi(x)\,\theta(z)\,dx\,dz.$$

Several useful properties that follow immediately from the definition are:

$$\mathcal{W}(\langle\left(\begin{smallmatrix}a & \\ & a\end{smallmatrix}\right),1\rangle g,s) = \mathcal{W}(g,s) \text{ for any } a \in F^\times\mathbb{A}^{\times 2}, \tag{6a}$$

$$\mathcal{W}(\langle\left(\begin{smallmatrix}a & b\\ & d\end{smallmatrix}\right),1\rangle g,s) = \psi(-b/d)\mathcal{W}(\langle\left(\begin{smallmatrix}a & \\ & d\end{smallmatrix}\right),1\rangle g,s), \tag{6b}$$

$$\mathcal{W}(gk,s) = \mathcal{W}(g,s) \text{ for } k \in K, \tag{6c}$$

$$\theta(z)^{-1}\psi(x)^{-1}\mathcal{W}(g,s) = \mathcal{W}(\langle\left(\begin{smallmatrix}z & \\ & z\end{smallmatrix}\right),\gamma(z)\rangle\langle\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right),1\rangle g,s). \tag{6d}$$

Relations (6a) through (6c) imply that

$$\mathcal{W}(g,s) = \alpha\,\mathcal{W}\big(\langle\left(\begin{smallmatrix}a\tau^{2+c} & \\ & \tau^c\end{smallmatrix}\right),1\rangle,s\big) \tag{6e}$$

where α and $a = \prod_v \tau_v^{e_v}$ depend on g. Then relations (6b) and (6c) lead to

$$\mathcal{W}(\langle\left(\begin{smallmatrix}a\tau^{2+c} & \\ & \tau^c\end{smallmatrix}\right),1\rangle,s) = 0 \text{ if } e_v < 0 \text{ for any } v. \tag{6f}$$

Furthermore, in the course of proving the Main Theorem we will show in Lemma 5 that

$$\mathcal{W}\big(\langle\left(\begin{smallmatrix}a\tau^{2+c} & \\ & \tau^c\end{smallmatrix}\right),1\rangle,s\big) = 0 \tag{7}$$

unless $\sum_v \deg(\pi_v)e_v \equiv 0 \pmod 2$.

Note that equations (6e) and (6f) contain the element $\tau = \tau_\infty$. In order to obtain a simple condition on the e_v we have included 2 in the exponent of τ. This is a reflection of the fact the the conductor of ψ_∞ is $\pi_\infty^2\mathfrak{O}_\infty$ while for finite v the conductor of ψ_v is simply $\mathfrak{O}_v$.

Equation (6d) shows that $\mathcal{W}(g,s)$ belongs to a space of functions called the Whittaker model as defined in Gelbart, Howe, and Piatetski-Shapiro [4]. Their work on the existence and uniqueness of the Whittaker model for the metaplectic group predicts the existence of an Euler product for $\mathcal{W}(g,s)$. In the next section we will explicitly give the Euler product.

Because our purpose is to study the Euler product and Riemann Hypothesis condition for $\mathcal{W}(g,s)$ the preceding discussion shows that it will suffice to assume that g is of the form $\langle(\begin{smallmatrix} a\tau^{2+c} & \\ & \tau^c \end{smallmatrix}),1\rangle$ where $a=\prod_v \tau_v^{e_v}$, $e_v \geq 0$ for all v, and $\sum_v \deg(\pi_v)e_v$ is even. Equation (6a) shows that c may always be taken to be 0 or 1. To avoid the matrix notation we will write

$$\mathcal{W}(a,c,s)=\mathcal{W}(\langle(\begin{smallmatrix} a\tau^{2+c} & \\ & \tau^c \end{smallmatrix}),1\rangle,s). \tag{8}$$

Lemma 3 (given below) will show that $\mathcal{W}(a,1,s)=\theta(\tau)\mathcal{W}(a,0,s)$. Therefore, it will be sufficient to study the Whittaker function $\mathcal{W}(a,0,s)$.

2.4. **The Main Theorem.** In this section we will state the Main Theorem. The normalized Whittaker function is

$$\hat{\mathcal{W}}(g,s)=\xi(2s)\,\mathcal{W}(g,s).$$

Recall that $\xi(s)$, $g_v(y)$, π_v, τ_v, q_v, x_v, and x were defined in Section 1.3. Let $a=\prod_v \tau_v^{e_v}$ be an idele with $e_v\geq 0$ for all v and $\hat{a}=\prod_{v<\infty}\pi_v^{e_v}$. Let b be the idele whose vth component is $b_v=\theta(\tau)^{\deg(\pi_v)}q_v^{-1/2}g_v(\pi_v)$. Then $b_v=\pm 1$. We define some local factors $w_v(a,s)$: If e_v is odd, then

$$w_v(a,s)=q_v^{-(e_v-1)/2}x_v^{-(e_v-1)/2}\big(1+x_v^2+\cdots+x_v^{e_v-1}\big),$$

and if e_v is even, then

$$w_v(a,s)=q_v^{e_v/2}x_v^{-e_v/2}\big(1-b_vq_v^{-1/2}x_v+x_v^2-b_vq_v^{-1/2}x_v^3+\cdots+x_v^{e_v}\big).$$

Denote by $L_{F(\sqrt{\hat{a}})/F}(s)$ the L-function associated with the field extension $F(\sqrt{\hat{a}})/F$:

$$L_{F(\sqrt{\hat{a}})/F}(s)=\prod_{v\notin X(\hat{a})}\big(1-(\pi_v,\hat{a})_vq_v^{-s}\big)^{-1}$$

where $X(\hat{a})=\{v\,|\,v(\hat{a})\equiv 1 \pmod 2\}$. In the event that $X(\hat{a})$ is empty then this L-function is just the zeta function of the projective line $\mathbb{P}^1(\mathbb{F}_q)$. Otherwise, it is the L-function of the hyperelliptic curve $y^2=\prod_{v\in S}\pi_v$ where $S=X(\hat{a})-\{\infty\}$ and each π_v is the monic irreducible polynomial in $\mathbb{F}_q[T]$ associated with the place v.

Now we may state the theorem:

Main Theorem. *Let $a=\prod_v\tau_v^{e_v}$. Suppose that $e_v\geq 0$ for all v, that $\sum_v\deg(\pi_v)e_v$ is even, and that $b_\infty=1$. Then the normalized Whittaker function has an Euler product which may be expressed as*

$$\hat{\mathcal{W}}(a,0,s)=x\prod_{v\in X(a)}q_v^{-1/2}x_v^{-1/2}\cdot\prod_v w_v(a,s)\cdot L_{F(\sqrt{\hat{a}})/F}(2s+\tfrac{1}{2}).$$

Also $\hat{\mathcal{W}}(a,0,s)$ satisfies the Riemann Hypothesis condition because all of its zeros are on the line $\Re(s)=0$.

If $e_v<0$ for some v or if $\sum_v\deg(\pi_v)e_v$ is odd, then $\hat{\mathcal{W}}(a,0,s)$ vanishes for all s. In the case that $b_\infty=-1$ then the Whittaker function is as above except that q^{-2s} must be replaced by $-q^{-2s}$ where $\hat{\mathcal{W}}(a,0,s)$ is regarded as *a power series in* q^{-2s}*. In this case the Riemann Hypothesis condition and Euler product still hold.*

Thus, the metaplectic Whittaker function $\mathcal{W}(a,0,s)$ satisfies all of the conditions of the Desired Theorem as stated in the Introduction!

From the general theory of Eisenstein series we already know that $\hat{\mathcal{W}}(a,0,s)$ has meromorphic equation and functional equation; however, the proof of the Main Theorem reveals exactly what the functional equation is. Hence, we state this as a corollary.

Corollary. $\hat{W}(a,0,s)$ *has meromorphic continuation and satisfies the functional equation:*

$$\hat{\mathcal{W}}(a,0,s) = \hat{\mathcal{W}}(a,0,-s).$$

3. Proof of the Main Theorem

We will prove the Main Theorem in three main steps. First, in Section 3.1 we will use the technique of Hecke operators to obtain the factors $w_v(a,s)$ in the Euler product. Second, in Section 3.2 we will obtain the L-function part of the Euler product. Third, in Section 3.3 we will establish the Riemann Hypothesis condition.

3.1. **The Euler Product I – The Factors** $w_v(a,s)$.

We will show that the Whittaker function has the required Euler product. Using Hecke operators we will prove Lemma 4 which will extract the local factors $w_v(a,s)$ mentioned in the Main Theorem. This will reduce the problem to the determination of $\mathcal{W}(a,c,s)$ when a is squarefree. We assume that $a = \prod_v \tau_v^{e_v}$ where $e_v \geq 0$ for all places v.

Let $\xi_{v,i}$ be coset representatives of the double coset

$$K\langle\left(\begin{smallmatrix}\tau_v^2 & \\ & 1\end{smallmatrix}\right),1\rangle K = \bigcup_i \xi_{v,i}K.$$

We define a Hecke operator T_v for each place v on the space of right K-invariant functions on $\tilde{G}$ by

$$(T_v f)(g) = \sum_i f(g\xi_{v,i}).$$

A complete set of coset representatives $\xi_{v,i}$ is given by the following three types of elements:

$$\langle\left(\begin{smallmatrix}\tau_v^2 & \alpha \\ & 1\end{smallmatrix}\right),1\rangle, \qquad \langle\left(\begin{smallmatrix}1 & \\ & \tau_v^2\end{smallmatrix}\right),1\rangle, \qquad \langle\left(\begin{smallmatrix}\tau_v & \beta \\ & \tau_v\end{smallmatrix}\right),(\beta_v,\pi_v)_v\rangle \tag{9}$$

where $\alpha_v \in \mathfrak{O}_v/\pi_v^2\mathfrak{O}_v$ and $\alpha_w = 0$ for $w \neq v$, and where $\beta_v \in \mathfrak{O}_v/\pi_v\mathfrak{O}_v$, $|\beta_v|_v = 1$, and $\beta_w = 0$ for $w \neq v$.

The subspace of $Ind_{\tilde{M}N}^{\tilde{G}}(\chi_s)$ that is K-fixed is one dimensional. Thus the spherical vector f_s is an eigenfunction of T_v. That is, $T_v f_s = \lambda_v f_s$. Because

$f_s(\langle\left(\begin{smallmatrix}1&\\&1\end{smallmatrix}\right),1\rangle) = 1$ we may compute the eigenvalue λ_v as follows:

$$\begin{aligned}\lambda_v &= \sum_i f_s(\xi_{v,i})\\ &= \sum_\alpha f_s(\langle\left(\begin{smallmatrix}\tau_v^2&\alpha\\&1\end{smallmatrix}\right),1\rangle) + f_s(\langle\left(\begin{smallmatrix}1&\\&\tau_v^2\end{smallmatrix}\right),1\rangle) + \sum_\beta f_s(\langle\left(\begin{smallmatrix}\tau_v&\beta\\&\tau_v\end{smallmatrix}\right),(\beta_v,\pi_v)_v\rangle)\\ &= q_v(x_v + x_v^{-1})\end{aligned}$$

where $x_v = q_v^{-2s}$.

Because the formation of both the pre-Whittaker function and the Whittaker function involve an operation on the left on f_s while the Hecke operator T_v acts on f_s on the right we obtain

$$\lambda_v\, W(g,s) = \sum_i W(g\xi_{v,i},s). \tag{10}$$

Note that (10) also holds for the Whittaker function $\mathcal{W}(a,c,s)$.

If $\ell \geq 0$ let

$$\Phi_v(\ell,s) = q_v^{-\ell}x_v^{-\ell}(1 + x_v^2 + \cdots + x_v^{2\ell}),$$

and if $\ell < 0$, let $\Phi_v(\ell,s) = 0$. Then for $\ell \geq 0$

$$\Phi_v(\ell,s)\,\Phi_v(1,s) - q_v^{-2}\Phi_v(\ell-1,s) = \Phi_v(\ell+1,s). \tag{11}$$

Lemma 1. *If $v(a) > 2$, then the recursion formula* (10) *takes the form*

$$W(a,c,s) = \Phi_v(1,s)\,W(a\tau_v^{-2},c,s) - q_v^{-2}\,W(a\tau_v^{-4},c,s). \tag{12}$$

If $v(a) = 2$, then (10) *becomes*

(13)

$$W(a,c,s) = \Phi_v(1,s)\,W(a\tau_v^{-2},c,s) - q_v^{-2}\,g_v(\pi_v)\,W(a\tau_v^{-2},c+\deg(\pi_v),s).$$

Proof. Let $A(e_v) = 1$ if $e_v \geq 0$ and $A(e_v) = 0$ is $e_v < 0$. Let $B(e_v) = 1$ if $e_v = 0$ and $B(e_v) = 0$ if $e_v \neq 0$. With $\xi_{v,i}$ as in (9), equation (10) becomes

$$\begin{aligned}&\lambda_v\, W(a,c,s)\\ &= \sum_\alpha W\big(\langle\left(\begin{smallmatrix}a\tau^{2+c}&\\&\tau^c\end{smallmatrix}\right),1\rangle\langle\left(\begin{smallmatrix}\tau_v^2&\alpha\\&1\end{smallmatrix}\right),1\rangle\big) + W\big(\langle\left(\begin{smallmatrix}a\tau^{2+c}&\\&\tau^c\end{smallmatrix}\right),1\rangle\langle\left(\begin{smallmatrix}1&\\&\tau_v^2\end{smallmatrix}\right),1\rangle\big)\\ &\quad + \sum_\beta W\big(\langle\left(\begin{smallmatrix}a\tau^{2+c}&\\&\tau^c\end{smallmatrix}\right),1\rangle\langle\left(\begin{smallmatrix}\tau_v&\beta\\&\tau_v\end{smallmatrix}\right),(\beta_v,\pi_v)_v\rangle\big)\\ &= \sum_\alpha \psi_v(-\alpha a\tau^2)\cdot W(a\tau_v^2,c,s) + W(a\tau_v^{-2})\\ &\quad + \sum_\beta (\beta_v,\pi_v)_v\psi_v(-\beta a\tau_v^{-1}\tau^2)\cdot W(a,c+\deg(\pi_v),s)\\ &= A(e_v)\,q_v^2\,W(a\tau_v^2,c,s) + W(a\tau_v^{-2},c,s)\\ &\quad + B(e_v)\,g_v(\pi_v)\,W(a,c+\deg(\pi_v),s).\end{aligned}$$

Solving the last equation for $W(a\tau_v^2,c,s)$ and replacing $a\tau_v^2$ with a gives the lemma. ∎

Lemma 2. *Suppose $\ell \geq 0$ and $v(a) = 0$. Then*

$$W(a\tau_v^{2\ell+1}, c, s) = \Phi_v(\ell, s) W(a\tau_v, c, s) \tag{14}$$

and

$$W(a\tau_v^{2\ell}, c, s) = \Phi_v(\ell, s)\, W(a, c, s) - q_v^2\, \Phi_v(\ell-1, s)\, g_v(\pi_v)\, W(a, c + \deg(\pi_v), s). \tag{15}$$

Proof. Suppose $d = 0$ or 1 and $\ell \geq 1$. Applying equation (12) from Lemma 1 repeatedly gives

$$\begin{aligned}
&W(a\tau_v^{2\ell+d}, c, s) \\
&= \Phi_v(1, s)\, W(a\tau_v^{2(\ell-1)+d}, c, s) - q_v^2\, \Phi_v(0, s)\, W(a\tau_v^{2(\ell-1)+d}, c, s) \\
&= \Phi_v(2, s)\, W(a\tau_v^{2(\ell-2)+d}, c, s) - q_v^2\, \Phi_v(1, s)\, W(a\tau_v^{2(\ell-3)+d}, c, s) \\
&\vdots \\
&= \Phi_v(\ell-1, s)\, W(a\tau_v^{2+d}, c, s) - q_v^2\, \Phi_v(\ell-2, s)\, W(a\tau_v^{d}, c, s)
\end{aligned}$$

If $d = 1$ we may apply equation (12) once more to obtain equation (14). Otherwise, we apply equation (13) to obtain equation (15). □

Lemma 3. *Let $g = \langle \mathrm{diag}(a\tau^{2+c}, \tau^c), 1\rangle$, and let θ be one of the two characters on Z_1/Z_0 determined by $\theta(z) = (a, z)$ for $z \in \prod_v \mathfrak{O}_v^\times$ and $\theta(\tau) = \pm 1$. Then the Whittaker function $\mathcal{W}(g, s) = \mathcal{W}(a, c, s)$ is the sum of two pre-Whittaker functions:*

$$\mathcal{W}(a, c, s) = W(a, c, s) + \theta(\tau)\, W(a, c+1, s).$$

If θ is any other character, then $\mathcal{W}(g, s)$ vanishes.

Proof.

$$\begin{aligned}
\mathcal{W}(g, s) &= \int_{\mathbb{A}^\times / F^\times \mathbb{A}^{\times 2}} W\left(\langle \left(\begin{smallmatrix} z & \\ & z \end{smallmatrix}\right), \gamma(z)\rangle g, s\right) \theta(z)\, dz \\
&= \sum_{j \in \{0,1\}} \int_{\prod_v \mathfrak{O}_v^\times / \prod_v \mathfrak{O}_v^{\times 2}} W\big(\langle \left(\begin{smallmatrix} z\tau^j & \\ & z\tau^j \end{smallmatrix}\right), \gamma(z\tau^j)\rangle g, s\big) \theta(z\tau^j)\, dz.
\end{aligned}$$

Because $\langle \mathrm{diag}(z, z), 1\rangle$ is in the maximal compact subgroup K factoring it to the right and removing it gives

$$\begin{aligned}
\mathcal{W}(g, s) = \sum_{j \in \{0,1\}} &\theta(\tau^j) W\big(\langle \left(\begin{smallmatrix} \tau^j & \\ & \tau^j \end{smallmatrix}\right), \gamma(\tau^j)\rangle g, s\big) \\
&\cdot \int_{\prod_v \mathfrak{O}_v^\times / \prod_v \mathfrak{O}_v^{\times 2}} \gamma(z)\, (a, z)\, \theta(z)\, dz.
\end{aligned}$$

As γ restricted to $\prod_v \mathfrak{O}_v^\times$ is trivial the integral in the last expression becomes

$$\int_{\prod_v \mathfrak{O}_v^\times / \prod_v \mathfrak{O}_v^{\times 2}} (a, z)\theta(z)\, dz.$$

This integral is zero unless θ restricted to $\prod_v \mathfrak{O}_v^\times$ is $\theta(z) = (a,z)$. There are only two such characters. For one of them $\theta(\tau) = 1$ while for the other $\theta(\tau) = -1$. Thus, the Whittaker function $\mathcal{W}(a,c,s)$ can be nonzero only if θ is one of these characters. Assuming θ is one of these characters and normalizing the measure so that $\mathrm{vol}(\prod_v \mathfrak{O}_v^\times / \prod_v \mathfrak{O}_v^{\times 2}) = 1$ gives the lemma. $\square$

Lemma 4. *Suppose $\ell \geq 0$ and $v(a) = 0$. Then*

$$\mathcal{W}(a\tau_v^{2\ell+1}, c, s) = w_v(a\tau^{2\ell+1}, s)\, \mathcal{W}(a\tau_v, c, s) \tag{16}$$

and

$$\mathcal{W}(a\tau_v^{2\ell}, c, s) = w_v(a\tau^{2\ell}, s)\, \mathcal{W}(a, c, s). \tag{17}$$

Proof. Recall that the local factor $w_v(a,s)$ was defined in Section 2.4. Equation (16) is immediate from Lemmas 2 and 3 because in this case $\Phi_v(\ell,s) = w_v(a\tau^{2\ell+1}, s)$. Equation (17) follows similarly:

$$\begin{aligned}
&\mathcal{W}(a\tau_v^{2\ell}, c, s)\\
&= \Phi_v(\ell,s)\, W(a,c,s) - q_v^2\, \Phi_v(\ell-1,s)\, g_v(\pi_v)\, W(a, c+\deg(\pi_v), s)\\
&\quad + \theta(\tau)\big\{\Phi_v(\ell,s)\, W(a,c+1,s) - q_v^2\, \Phi_v(\ell-1,s)\, g_v(\pi_v)\, W(a, c+1+\deg(\pi_v), s)\big\}\\
&= \Phi_v(\ell,s)\, \mathcal{W}(a,c,s) - q_v^2\, \Phi_v(\ell-1,s)\, g_v(\pi_v)\, \mathcal{W}(a, c+\deg(\pi_v), s)\\
&= \big\{\Phi_v(\ell,s) - \theta(\tau)^{\deg(\pi_v)} q_v^2\, g_v(\pi_v)\, \Phi_v(\ell-1,s)\big\}\mathcal{W}(a,c,s)\\
&= w_v(a\tau^{2\ell}, s)\, \mathcal{W}(a,c,s).
\end{aligned}$$

$\square$

Lemma 4 shows that the question of the Riemann Hypothesis condition for any Whittaker function $\mathcal{W}(a,g,s)$ with $a = \prod_v \tau_v^{e_v}$ is reduce to the same question for the local factors $w_v(a,s)$ and for $\mathcal{W}(a,g,s)$ where $e_v = v(a)$ is 0 or 1 for all places v. Thus we will need to evaluate the Whittaker function only at squarefree elements of the monoid generated by the τ_v. This will be accomplished in the next section.

3.2. **The Euler Product II – The L-function.** We will now convert the integral for the pre-Whittaker function into a Dirichlet series in Lemmas 5 through 8 below. Then Lemma 9 will express the Whittaker function in terms of the L-function of a hyperelliptic curve.

For a finite set S of places define $h_S(a)$ to be

$$h_S(a) = \prod_{v \in S - \{\infty\}} \pi_v^{e_v+1}.$$

Lemma 5. *Let $a = \prod_v \tau_v^{e_v}$ where $0 \leq e_v \leq 1$ for all v. Let $r = \sum_v \deg(\pi_v) e_v$, and suppose r is even. Then*

$$W(a,c,s) = q|a\tau^2|^{1/2-s} \sum_S^{(c)} \prod_{v\in S} (h_S(a), a\pi_v)_v\, g_v(\pi_v^{e_v+1})\, q_v^{-1-2s(e_v+1)} \tag{18}$$

where the summation is over finite sets S of places v but where the (c) indicates that each set S must satisfy

$$\sum_{v\in S}\deg(\pi_v)(e_v+1)\equiv c \pmod 2.$$

If r is odd, then the pre-Whittaker function vanishes.

Proof. Let $\hat{a}=\prod_{v<\infty}\pi_v^{e_v}$. Then $\hat{a}\in\mathbb{F}_q[T]$ and $a=\hat{a}\tau^r$. We have

$$\begin{aligned}
W(a,c,s) &= \int_{\mathbb{A}} f_s\big(\langle\left(\begin{smallmatrix}0&1\\1&x\end{smallmatrix}\right),1\rangle\langle\left(\begin{smallmatrix}\hat{a}\tau^{2+r+c}&0\\0&\tau^c\end{smallmatrix}\right),1\rangle\big)\,\psi(x)\,dx\\
&= \int_{\mathbb{A}} f_s\big(\langle\left(\begin{smallmatrix}1&0\\0&\hat{a}\end{smallmatrix}\right),1\rangle\langle\left(\begin{smallmatrix}\tau^c&0\\0&\tau^{2+r+c}\end{smallmatrix}\right),1\rangle\langle\left(\begin{smallmatrix}0&1\\1&a^{-1}\tau^{-2}x\end{smallmatrix}\right),1\rangle\big)\,\psi(x)\,dx\\
&= \int_{\mathbb{A}} f_s\big(\langle\left(\begin{smallmatrix}\tau^c&0\\0&\tau^{2+r+c}\end{smallmatrix}\right),1\rangle\langle\left(\begin{smallmatrix}0&1\\1&a^{-1}\tau^{-2}x\end{smallmatrix}\right),1\rangle\big)\,\psi(x)\,dx
\end{aligned}$$

which by the intertwining operator (5) is

$$=\sum_{h\in H_F^2\backslash H_F}\tfrac{1}{4}\int_{\mathbb{A}} f'_s\big(h\langle\left(\begin{smallmatrix}\tau^c&0\\0&\tau^{2+r+c}\end{smallmatrix}\right),1\rangle\langle\left(\begin{smallmatrix}0&1\\1&a^{-1}\tau^{-2}x\end{smallmatrix}\right),1\rangle\big)\,\psi(x)\,dx.$$

For a fixed h consider the integral in the last expression. Because f'_s is right invariant by K we may insert an element $\langle\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right),1\rangle$ with $\alpha\in\prod_v\mathfrak{O}_v^\times$. We pass it to the left and remove it using the fact that f'_s is in $Ind_{\tilde{T}_aN}^{\tilde{G}}(\xi_s)$. This produces an extra Hilbert symbol that must be trivial unless the integral vanishes. That is, if $h=\langle\left(\begin{smallmatrix}h_1&0\\0&h_2\end{smallmatrix}\right),1\rangle$ then the integral is zero unless

$$(h_1h_2\tau^r,\alpha)=1$$

for all α in $\prod_v\mathfrak{O}_v^\times$. Thus, assuming the integral is not zero forces the condition that r is even and allows us to suppose $h_1=bh_2^{-1}$ where b is in $\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}$. Thus

$$W(a,c,s)=\sum_{h\in\mathcal{H}}\prod_v W_v(h,a,c,s) \tag{19}$$

where $\mathcal{H}$ is a set of monic representatives of polynomials in $\mathbb{F}_q[T]$ modulo squared polynomials and where

$$W_v(h,a,c,s)=\int_{F_v} f'_{v,s}\big(\langle\left(\begin{smallmatrix}h\tau^c&0\\0&h^{-1}\tau^{2+r+c}\end{smallmatrix}\right),1\rangle\langle\left(\begin{smallmatrix}0&1\\1&a^{-1}\tau^{-2}x\end{smallmatrix}\right),1\rangle\big)\,\psi_v(x)\,dx.$$

We may evaluate $f'_{v,s}$ in the integrand by upper triangularizing its argument with an appropriately chosen element of $K_v=GL_2(\mathfrak{O}_v)$. If $|x|_v\le|a\tau^2|_v$, then (considering only the vth components)

$$\langle\left(\begin{smallmatrix}0&1\\1&a^{-1}\tau^{-2}x\end{smallmatrix}\right),1\rangle\underbrace{\langle\left(\begin{smallmatrix}-a^{-1}\tau^{-2}x&1\\1&0\end{smallmatrix}\right),1\rangle}_{\in K_v}=\langle\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),1\rangle, \tag{20}$$

and if $|x|_v > |a\tau^2|_v$,

$$\langle(\begin{smallmatrix} 0 & 1 \\ 1 & a^{-1}\tau^{-2}x \end{smallmatrix}), 1\rangle \underbrace{\langle(\begin{smallmatrix} 1 & 0 \\ -a\tau^2x^{-1} & 1 \end{smallmatrix}), 1\rangle}_{\in K_v} = \langle(\begin{smallmatrix} -a\tau^2x^{-1} & 1 \\ 0 & a^{-1}\tau^{-2}x \end{smallmatrix}), 1\rangle. \tag{21}$$

Because of (20) and (21) it is natural to split the local integral as

$$W_v(h,a,c,s) = I_v(h,a,c,s) + II_v(h,a,c,s)$$

where $I_v(h,a,c,s)$ is the integral on the set $|x|_v \le |a\tau^2|_v$ and $II_v(h,a,c,s)$ is the integral on the set $|x|_v > |a\tau^2|_v$. Then

$$W(a,c,s) = \sum_{\substack{h\in\mathcal{H} \\ S}} \prod_{v\notin S} I_v(h,a,c,s) \prod_{v\in S} II_v(h,a,c,s). \tag{22}$$

where the summation is over $h \in \mathcal{H}$ and finite sets S of places v.

Let

$$\epsilon_v(x) = \begin{cases} 1 & \text{if } v(x) \text{ is even,} \\ 0 & \text{if } v(x) \text{ is odd.} \end{cases}$$

Because $\psi_v(x)$ is trivial on $|x|_v \le |a\tau^2|_v$ we obtain

$$\begin{aligned} I_v(h,a,c,s) &= \int_{|x|\le|a\tau^2|} f_{v,s}\big(\langle(\begin{smallmatrix} h\tau^c & 0 \\ 0 & h^{-1}\tau^{2+r+c} \end{smallmatrix}), 1\rangle\big)\, \psi_v(x)\, dx \\ &= \epsilon_v(h\tau^c)\, |h|_v^{2s+1}\, |\tau^{-r-2}|_v^{s+1/2}\, |a\tau^2|_v \operatorname{vol}(\mathfrak{O}_v). \end{aligned} \tag{23}$$

Now, $II_v(h,a,c,s)$ is the integral

$$\int_{|x|_v>|a\tau^2|} f'_{v,s}\big(\langle(\begin{smallmatrix} h\tau^c & 0 \\ 0 & h^{-1}\tau^{2+r+c} \end{smallmatrix}), 1\rangle\langle(\begin{smallmatrix} -a\tau^2x^{-1} & 1 \\ 0 & a^{-1}\tau^{-2}x \end{smallmatrix}), 1\rangle\big)\, \psi_v(x)\, dx.$$

Due to the additive character the integral vanishes on the set $|x|_v > q_v|\tau^2|_v$. Then because $v(a)$ is 0 or 1 it is only necessary to integrate on the set $|x|_v = q_v|\tau^2|_v$. We obtain

(24) $II_v(h,a,c,s) =$

$$\epsilon_v(ha\pi_v\tau^c)\, |h|_v^{2s+1}\, |\tau^{-2-r}|_v^{s+1/2}\, |a\pi_v|_v^{2s+1}\, (h, a\pi_v)_v\, g_v(a\pi_v)\, |\tau^2|_v \operatorname{vol}(\mathfrak{O}_v).$$

According to equations (23) and (24) the products in equation (22) will vanish unless certain conditions are met. For a given S there is exactly one h such that

$$\prod_{v\notin S\cup\{\infty\}} \epsilon_v(h\tau^c) \prod_{v\in S-\{\infty\}} \epsilon_v(ha\pi_v\tau^c) = 1. \tag{25}$$

The h corresponding to S is

$$h_S(a) = \prod_{v\in S-\{\infty\}} \pi_v^{e_v+1}.$$

Thus, we may simply sum over finite sets S with the corresponding h_S. However, including the term corresponding to the infinite place in (25) gives a constraint on S:

$$\sum_{v \in S} \deg(\pi_v)(e_v + 1) \equiv c \pmod 2.$$

Therefore we may re-index the summation as

$$W(a,c,s) = \sum_S^{(c)} \prod_{v \notin S} I_v(h_S(a), a, c, s) \prod_{v \in S} II_v(h_S(a), a, c, s) \tag{26}$$

where the (c) on the summation indicates that the summation is taken over all finite sets S of places such that $\sum_{v \in S} \deg(\pi_v)(e_v+1) \equiv c \pmod 2$. Note that the expression for $W(a,c,s)$ is almost an Euler product.

Next, by using the explicit formulas (23) and (24) for $I_v(h,a,c,s)$ and $II_v(h,a,c,s)$ we may reduce equation (26) to

$$W(a,c,s) = q|a\tau^2|^{1/2-s} \sum_S^{(c)} \prod_{v \in S} (h_S(a), a\pi_v)_v \, g_v(a\pi_v) \, q_v^{-1-2s(e_v+1)}.$$

This completes the proof of Lemma 5. $\square$

It will be important to simplify the products of Hilbert symbols and Gauss sums that appear in Lemma 5. To do this we will use the functional equation of the Whittaker function. This is given in the next lemma. As with the pre-Whittaker function, the pre-constant term $C(g,s)$ is a multiple of $C(\langle \begin{pmatrix} a\tau^{2+c} & \\ & \tau^c \end{pmatrix}, 1\rangle, s)$ for some $a = \prod_v \tau_v^{e_v}$ and c. Let

$$\xi(s) = q^{-2s}(1-q^{-2s})^{-1}(1-q^{1-2s})^{-1},$$

and let the normalized pre-constant term be $\hat{C}(g,s) = \xi(2s)C(g,s)$.

Lemma 6. *Let $a = \prod_v \tau_v^{e_v}$ and $r = \sum_v \deg(\pi_v)e_v$. If r and c are both even then*

$$\hat{C}(\langle \begin{pmatrix} a\tau^{2+c} & \\ & \tau^c \end{pmatrix}, 1\rangle, s) = q^{(2+r)(-s-1/2)}\xi(2s) + q^{(2+r)(s-1/2)}\xi(-2s).$$

If either r or c is odd, then $\hat{C}(a,c,s) = 0$. Thus there is a functional equation

$$\hat{C}(g,s) = \hat{C}(g,-s)$$

which by the general theory of Eisenstein series implies that

$$\hat{W}(g,s) = \hat{W}(g,-s).$$

Proof. The integral for the pre-constant term is similar to the integral for the pre-Whittaker function. However, the lack of the additive character makes the computation simpler than the one in Lemma 5. For brevity we omit the details. The fact that the pre-Whittaker functions satisfy the same functional equation as the constant term is explained in Moeglin and Waldspurger [16] which has become a standard reference on the theory of Eisenstein series. Their work is heavily influenced by the seminal work of Langlands [13] on Eisenstein series. $\square$

We are now ready to simplify the product of Gauss sums and Hilbert symbols in the Dirichlet series for $W(a,c,s)$ in Lemma 5. The next two lemmas will accomplish this.

For a finite set of places the degree of S, $\deg(S)$, is defined to be

$$\deg(S) = \sum_{v\in S} \deg(\pi_v).$$

Lemma 7. *Let X denote the set of places for which $e_v = v(a)$ is odd, let S be a finite set of places, and let*

$$P(S) = \prod_{v\in S} (h_S(a), a\pi_v)_v g_v(\pi_v^{e_v+1}).$$

Then if $\deg(S-X)$ *is even*

$$P(S) = \prod_{v\in S\cap X} (-1) \prod_{v\in S-X} (\pi_v, a)_v q_v^{1/2}$$

while if $\deg(S-X)$ *is odd*

$$P(S) = \prod_{v\in S\cap X} (-1) \prod_{v\in S-X} (\pi_v, a)_v q_v^{1/2} \cdot g_\infty(\pi_\infty) q^{-1/2}.$$

Proof. We will begin by first proving a special case. Let

$$b(S) = \prod_{v\in S} (h_S(1), \pi_v)_v\, g_v(\pi_v)\, q_v^{-1/2}.$$

Then we will show

$$b(S) = \begin{cases} 1 & \text{if } \deg(S) \equiv 0, \\ g_\infty(\pi_\infty)\, q^{-1/2} & \text{if } \deg(S) \equiv 1. \end{cases} \tag{27}$$

The key to proving equation (27) is to consider the pre-Whittaker functions $W(1,0,s)$ and $W(1,1,s)$. In this case $e_v = 0$ for all v allowing the pre-Whittaker function to be expressed in terms of $b(S)$ as follows:

$$W(1,0,s) = q^{2s} \sum_{i=0}^{\infty} c_i q^{-2si}$$

where if i is odd $c_i = 0$ and if i is even

$$c_i = \sum_{\deg(S)=i} b(S) q^{-\deg(S)/2}.$$

Now let $\alpha(s) = q^{2s}(1-q^{1-4s})$. Then the functional equation $\xi(2s)W(g,s) = \xi(-2s)W(g,-s)$ from Lemma 6 implies $\alpha(s)W(g,s) = -\alpha(-s)W(g,-s)$. The advantage of this second functional equation is that $\alpha(s)W(g,s)$ has no poles. This is because the order of a pole of a pre-Whittaker function cannot exceed the order of a pole of the pre-constant term and $\alpha(s)C(g,s)$ has no pole. Writing $x = q^{-2s}$ we have

$$\alpha(s)W(1,0,s) = c_0x^{-2} + (c_2 - qc_0) + (c_4 - qc_2)x^2 + \cdots \tag{28}$$

which by the functional equation is also equal to

$$-c_0x^2 - (c_2 - qc_0) - (c_4 - qc_2)x^{-2} - \cdots . \tag{29}$$

Equating coefficients from (28) and (29) gives

$$\sum_{\deg(S)=2i} b(S) = \begin{cases} q^2 & \text{if } i = 1, \\ q^{2i} - q^{2i-2} & \text{if } i > 1. \end{cases} \tag{30}$$

Now we will show there is no cancellation in the summation involving $b(S)$. Each set S such that $\deg(S) = 2i$ will contribute to the sum. The number of such sets is the coefficient of x^{2i} in the series

$$\prod_v (1 + x^{\deg(\pi_v)}) = 1 + (q + 1)x + q^2x^2 + (q^3 - q)x^3 + (q^4 - q^2)x^4 + \cdots .$$

There are q^2 distinct sets of degree 2 and $q^{2i} - q^{2i-2}$ distinct sets of degree $2i$. Because $|b(S)| = 1$ there can be no cancellation in equation (30), which implies that if $\deg(S)$ is even then $b(S) = 1$. This proves half of the the special case given in equation (27). A similar argument with the pre-Whittaker function $W(1, 1, s)$ gives the other half.

Now consider

$$P(S) = \prod_{v\in S} (h_S(a), a\pi_v)_v g_v(\pi_v^{e_v+1}).$$

Squared factors of $h_S(a) = \prod_{v<\infty} \pi_v^{e_v+1}$ may be removed without affecting $P(S)$. This gives

$$P(S) = \prod_{v\in S\cap X} (h_{S-X}(1), a\pi_v)_v g_v(\pi_v^{e_v+1}) \prod_{v\in S-X} (h_{S-X}(1), a\pi_v)_v g_v(\pi_v^{e_v+1}).$$

For $v \in S \cap X$, $(h_{S-X}, a\pi_v)_v = 1$ and $g_v(\pi_v^{e_v+1}) = -1$. Thus

$$P(S) = \prod_{v\in S\cap X} (-1) \prod_{v\in S-X} (h_S(1), a\pi_v)_v \cdot b(S - X)q^{deg(S-X)}.$$

Then because $\prod_{v\in S-X}(h_{S-X}(1), a)_v = 1$ and because we already found the value of $b(S - X)$ in equation (27) we obtain the lemma. □

Lemma 8. *Suppose $a = \prod_v \tau_v^{e_v}$ where $0 \le e_v \le 1$, let $x_v = q_v^{-2s}$, and let X be the finite set of places for which $e_v = 1$. The pre-Whittaker function $W(a, c, s)$ may be written as*

$$W(a, c, s) = \alpha_c\, q|a\tau^2|^{1/2-s} \sum_{\substack{S \\ \deg(S-X)\equiv c}} \prod_{v\in S\cap X} (-1) \prod_{v\in S-X} (\pi_v, a)_v q_v^{1/2} \prod_{v\in S} q_v^{-1} x_v^{e_v+1}$$

where the summation is over all finite sets S such that $\deg(S - X)$ *is congruent to c modulo 2 and where*

$$\alpha_c = \begin{cases} 1 & \textit{if } c \textit{ is even}, \\ g_\infty(\pi_\infty)q^{-1/2} & \textit{if } c \textit{ is odd}. \end{cases}$$

Proof. Summing over finite sets S such that $\deg(S) \equiv c$ is equivalent to summing over finite sets S such that $\deg(S-X) \equiv c$. Combining the results of Lemmas 5 and 7 gives the lemma. $\square$

Suppose $a = \prod_v \tau_v^{e_v}$ and that $0 \leq e_v \leq 1$ for all v. Also suppose that $r = \sum_v \deg(\pi_v)e_v$ is even. Then the normalized Whittaker function $\hat{\mathcal{W}}(a,0,s)$ is an L-function. $\hat{\mathcal{W}}(a,1,s)$ is also an L-function because $\hat{\mathcal{W}}(a,0,s) = \theta(\tau)\hat{\mathcal{W}}(a,1,s)$. More precisely we have following lemma.

Lemma 9. *With the assumptions given above, if $\theta(\tau)g_\infty(\pi_\infty)q^{-1/2} = 1$, then*

$$\begin{aligned}\hat{\mathcal{W}}(a,0,s) &= q|a\tau^2|^{1/2-s}\prod_{v\notin X}(1-(\pi_v,\hat{a})_v q_v^{-1/2-2s})^{-1}\\ &= q|a\tau^2|^{1/2-s}L_{F(\hat{a})/F}(1/2+2s).\end{aligned}$$

If $\theta(\tau)g_\infty(\pi_\infty)q^{-1/2} = -1$, then the Whittaker function is as above except that q^{-2s} must be replaced q^{-2s} by $-q^{-2s}$ where the expression above is regarded as a power series in q^{-2s}.

Proof. Let $\beta = \theta(\tau)g_\infty(\pi_\infty)q^{-1/2}$. Then $\beta = \pm 1$. We will write $y = \beta x = \beta q^{-2s}$ and $y_v = y^{\deg(\pi_v)}$. X will mean $X(a)$. Then by Lemma 3 we have

$$\mathcal{W}(a,0,s) = W(a,0,s) + \theta(\tau)W(a,1,s)$$

and then by Lemma 8 this is

$$\begin{aligned}\alpha_c\, q|a\tau^2|^{1/2-s}\Big\{ &\sum_{\substack{S\\ \deg(S-X)\equiv 0}}\ \prod_{v\in S\cap X}(-1)\prod_{v\in S-X}(\pi_v,a)_v q_v^{1/2}\prod_{v\in S} q_v^{-1}x_v^{e_v+1}\\ &+\beta\sum_{\substack{S\\ \deg(S-X)\equiv 1}}\ \prod_{v\in S\cap X}(-1)\prod_{v\in S-X}(\pi_v,a)_v q_v^{1/2}\prod_{v\in S} q_v^{-1}x_v^{e_v+1}\Big\}.\end{aligned}$$

The two summations combine giving

$$\begin{aligned}&\alpha_c\, q|a\tau^2|^{1/2-s}\sum_S\prod_{v\in S\cap X}(-1)\prod_{v\in S-X}(\pi_v,a)_v q_v^{1/2}\prod_{v\in S}q_v^{-1}y_v^{e_v+1}\\ &= \alpha_c\, q|a\tau^2|^{1/2-s}\prod_{v\in X}\{1-q_v^{-1}y_v^{e_v+1}\}\prod_{v\notin X}\{1+(\pi_v,a)_v q_v^{-1/2}y_v^{e_v+1}\}\\ &= \alpha_c\, q|a\tau^2|^{1/2-s}\prod_{v}\{1-q_v^{-1}x_v^{e_v+1}\}\prod_{v\notin X}\left(1-(\pi_v,a)_v q_v^{-1/2}y_v^{e_v+1}\right)^{-1}\\ &= \alpha_c\, q|a\tau^2|^{1/2-s}\xi(2s)^{-1}\prod_{v\notin X}\left(1-(\pi_v,a)_v q_v^{-1/2}y_v^{e_v+1}\right)^{-1}\end{aligned}$$

If $\beta = 1$ then the product in the last expression is the desired L-function. If $\beta = -1$, then it is the L-function with q^{-2s} replaced by $-q^{-2s}$. This completes the proof of the lemma. $\square$

Lemmas 4 and 9 show that the Whittaker function has the Euler product given in the Main Theorem. Now we turn to the Riemann Hypothesis condition.

3.3. **The Riemann Hypothesis Condition.**

We must verify that the zeros of the Whittaker function lie on the line $\Re(s) = 0$. First consider the local factor $w_v(a, s)$. We will need the following lemma.

Lemma 10. *Let $r(z) = a_0 + a_1 z + \cdots + a_m z^m$ where the coefficients are complex. Let $\bar{r}(z)$ represent the polynomial obtained by replacing each coefficient of $r(z)$ by its conjugate, and define $r^*(z) = z^m \bar{r}(z^{-1})$. If all the zeros of $r(z)$ are inside the unit circle $|z| < 1$ and $|\gamma| = 1$, then all the zeros of $r(z) + \gamma r^*(z)$ are on the unit circle $|z| = 1$.*

Proof. This fact is found in the excellent book of Pólya and Szegö [18]. If $\alpha_1, \ldots, \alpha_m$ are the zeros of $r(z)$, then we may write

$$r(z) = a_m (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_m),$$
$$r^*(z) = \bar{a}_m (1 - \bar{\alpha}_1 z)(1 - \bar{\alpha}_2 z) \cdots (1 - \bar{\alpha}_m z).$$

If $r(z) + \gamma r^*(z) = 0$ for some z, then

$$\prod_{i=1}^{m} |z - \alpha_i| = \prod_{i=1}^{m} |1 - \bar{\alpha}_i z|.$$

If $|z| < 1$, then $|z - \alpha_i| < |1 - \bar{\alpha}_i z|$ for all i. Similarly, if $|z| > 1$, then $|z - \alpha_i| > |1 - \bar{\alpha}_i z|$ for all i. Hence, $|z| = 1$. □

Now we can deal with the local factors $w_v(a, s)$.

Lemma 11. *The factors $w_v(a, s)$ satisfy the Riemann Hypothesis condition.*

Proof. Recall that $a = \prod_v \tau_v^{e_v}$ where $e_v \geq 0$ for all e_v. In the case that e_v is odd $w_v(a, s)$ is a power of q_v^{-2s} times a cyclotomic polynomial in q_v^{-4s} which is zero only if $\Re(s) = 0$.

The case in which e_v is even requires lemma 10. Consider the polynomial

$$(1 - x^2)(1 - q^{-1/2} x + x^2 - q^{-1/2} x^3 - \cdots + x^e) = p(x) - x^{e+2} p(x^{-1})$$

where

$$p(x) = q^{-1/2} x^{e+1} - x^{e+2}.$$

All the zeros of $p(x)$ have absolute value less than 1. By lemma 10 the zeros of $p(x) - x^{e+1} p(x^{-1})$ have absolute value 1. Replace x by $\pm q_v^{-2s}$ and e by e_v. Then when e_v is even the zeros of $w_v(a, s)$ are on the line $\Re(s) = 0$.

□

Lemma 12. *$L_{F(\sqrt{a})/F}(2s + \frac{1}{2})$ satisfies the Riemann Hypothesis condition.*

Proof. Weil's celebrated theorem [21] for the zeta functions of curves over finite fields says that the zeros of $L_{F(\sqrt{a})/F}(s)$ are on the line $\Re(s) = 1/2$. Thus the zeros of $L_{F(\sqrt{a})/F}(2s + \frac{1}{2})$ are on the line $\Re(s) = 0$. For discussions of the Riemann Hypothesis for zeta functions of curves see [7] or [14]. □

Lemmas 11 and 12 show that the Whittaker function satisfies the Riemann Hypothesis condition. This completes the proof of the Main Theorem.

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