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Sums of exponential functions having only real zeros

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Abstract. Let $H_n(z)$ be the function of a complex variable z defined by

$$H_n(z) = \sum G(\pm ia_1 \pm \cdots \pm ia_n) e^{iz(\pm b_1 \pm \cdots \pm b_n)}$$

where the summation is over all 2^n possible plus and minus sign combinations, the same sign combination being used in both the argument of G and in the exponent. The numbers $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$ are assumed to be positive, and G is an entire function of genus 0 or 1 that is real on the real axis and has only real zeros. Then the function $H_n(z)$ has only real zeros.

1. Introduction

Let $G(z)$ be an entire function of genus 0 or 1 that is real on the real axis and has only real zeros. This is equivalent to saying that $G(z)$ has a Weierstrass product representation of the form $G(z) = cz^q e^{\alpha z} \prod (1 - z/\alpha_m)\, e^{z/\alpha_m}$ where q is a nonnegative integer, c and α are real, and the α_m are the nonzero real zeros of G. In the case of either genus 0 or 1, the sum $\sum \alpha_m^{-2}$ is finite. Let $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$ be sequences of positive real numbers, and let $H_n(w)$ be the function of a complex variable w defined by

$$H_n(w) = \sum G(\pm ia_1 \pm \cdots \pm ia_n) e^{iw(\pm b_1 \pm \cdots \pm b_n)}$$

where the summation is over all 2^n possible plus and minus sign combinations, the same sign combination being used in both the argument of G and in the exponent. For example,

$$\begin{aligned} H_2(w) = G(-ia_1 - ia_2)e^{iw(-b_1-b_2)} + G(-ia_1 + ia_2)e^{iw(-b_1+b_2)} \\ + G(ia_1 - ia_2)e^{iw(b_1-b_2)} + G(ia_1 + ia_2)e^{iw(b_1+b_2)}. \end{aligned}$$

In this paper we prove the following:

Theorem 1 *All of the zeros of $H_n(w)$ are real.*

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The proof of Theorem 1 is given in §2. An interesting corollary produces polynomials with zeros on the unit circle. In the special case $b_1 = b_2 = \cdots = b_n$ replacing $e^{ib_k w}$ with x gives a rational function whose numerator is an even polynomial. Replacing x^2 in the numerator with t gives a polynomial $P_n(t)$ of degree n in the variable t. For example, $H_2(w)$ becomes

$$G(-ia_1 - ia_2)x^{-2} + G(-ia_1 + ia_2) + G(ia_1 - ia_2) + G(ia_1 + ia_2)x^2$$

which becomes

$$P_2(t) = G(-ia_1 - ia_2) + G(-ia_1 + ia_2)t + G(ia_1 - ia_2)t + G(ia_1 + ia_2)t^2.$$

Because the zeros of $H_2(w)$ are real, the zeros of $P_2(t)$ are on the unit circle.

In other words, we may define

$$P_n(t) = \sum_{\sigma} G(\sigma \cdot (ia_1, \ldots, ia_n))t^{|\sigma|}$$

where the summation is over all 2^n vectors σ of the form $(\pm 1, \ldots, \pm 1)$, $|\sigma|$ represents the number of plus signs in the vector σ, and $\sigma \cdot (ia_1, \ldots, ia_n)$ is the ordinary dot product. Then an immediate corollary to Theorem 1 is the following 'circle theorem':

Theorem 2 *All the zeros of the polynomial $P_n(t)$ lie on the unit circle in the complex plane.*

Our main reason for studying sums of exponential functions with real zeros has to do with the Riemann Hypothesis. For $\Re(s) > 1$, the Riemann zeta function is defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. It has an analytic continuation and the function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

is entire. The Riemann Hypothesis states that all the zeros of $\xi(s)$ satisfy $\Re(s) = 1/2$. A proof of the Riemann Hypothesis would be a major advance for analytic number theory. Let $\Xi(z) = \xi\left(\frac{1}{2} + iz\right)$. It is well known (see Titchmarsh [8] chapter 10) that

$$\boldsymbol{\Xi}(z) = \int_{-\infty}^{\infty} \boldsymbol{\Phi}(x)e^{izx}dx$$

where

$$\boldsymbol{\Phi}(x) = \sum_{n=1}^{\infty} \left(4n^4\pi^2 e^{9x/2} - 6n^2\pi e^{5x/2}\right) \exp\left(-n^2\pi e^{2x}\right).$$

In other words, the Riemann Hypothesis is true if and only if the Fourier transform $\Xi(z)$ has only real zeros. In the process of constructing certain Fourier transforms with real zeros we found the result given in Theorem 1. In a different paper [2] we apply Theorem 1 to construct Fourier transforms with real zeros.

The proof of Theorem 1 involves blending an idea of Pólya with a method of Lee and Yang. In 1926 Pólya was attempting to understand the Riemann zeta function. His paper includes the following observation:

Proposition 3 (Pólya [5], Hilfssatz II) *Let $a > 0$ and let b be real. Assume $G(z)$ is an entire function of genus 0 or 1 that for real z takes real values, has at least one real zero, and has only real zeros. Then the function $G(z+ia)e^{ib}+G(z-ia)e^{-ib}$ has only real zeros.*

In 1952 Lee and Yang exhibited a model of phase transitions in lattice gases with the property that the zeros of the partition function for the system lie on the unit circle in the complex plane. The main mathematical result is (as re-stated in [7, p.108]):

Proposition 4 (Lee-Yang [4], Appendix II) *Let $(A_{ij})_{j\neq i}$ be a family of real numbers such that $-1 < A_{ij} < 1$, $A_{ij} = A_{ji}$ for $i = 1,\ldots,n$; $j = 1,\ldots,n$. We define a polynomial $\mathcal{P}_n$ in n variables by*

$$\mathcal{P}_n(z_1,\ldots,z_n) = \sum_S z^S \left(\prod_{i\in S}\prod_{j\in S'} A_{ij}\right)$$

where the summation is over all subsets $S = \{i_1,\ldots,i_s\}$ of $\{1,\ldots,n\}$, $z^S = z_{i_1}\cdots z_{i_s}$ and $S' = \{j_1,\ldots,j_{n-s}\}$ is the complement of S in $\{1,\ldots,n\}$. Then $\mathcal{P}_n(z_1,\ldots,z_n) = 0$ and $|z_1| \geq 1,\ldots,|z_{n-1}| \geq 1$ imply $|z_n| \leq 1$.

By setting $z_1,\ldots,z_n$ all equal to t one obtains the celebrated Lee-Yang Circle Theorem from statistical mechanics:

Corollary 5 *Let $\mathcal{P}_n(t) = \mathcal{P}_n(t,\ldots,t)$. All of the roots of $\mathcal{P}_n(t)$ lie on the unit circle.*

Although the exponential sums $H_n(z)$ and the related polynomials $P_n(t)$ do not immediately appear to have an interpretation related to phase transitions, the proof of the location of their zeros bears a strong resemblance the proof of Proposition 4 by Lee and Yang. The main step in our argument involves proving a result similar to Proposition 4 about multivariable polynomials (Lemma 8 below). Essentially, we use the 'decoupling method' of Lee and Yang in a way that takes into account Pólya's Hilfssatz II.

Given sequences of real numbers $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ it seems natural to iteratively apply the process of Proposition 3 (Pólya's Hilfssatz II) to form a sequence of functions $H_k(z,w)$ of complex variables z and w as follows:

$$H_0(z,w) = G(z), \quad \text{and}$$
$$H_k(z,w) = H_{k-1}(z-ia_k,w)e^{-ib_k w} + H_{k-1}(z+ia_k,w)e^{ib_k w} \quad \text{for} \quad k > 0.$$

For example,

$$\begin{aligned} H_2(z,w) = {} & G(z-ia_1-ia_2)e^{iw(-b_1-b_2)} + G(z+ia_1-ia_2)e^{iw(b_1-b_2)} \\ & + G(z-ia_1+ia_2)e^{iw(-b_1+b_2)} + G(z+ia_1+ia_2)e^{iw(b_1+b_2)}. \end{aligned}$$

In light of Proposition 3 it is evident that the functions $H_k(z,w_0)$ for fixed real w_0, if nonzero, are of genus 0 or 1, have only real zeros, and are real for real z.

One asks if is possible to obtain such functions as limits of this type of process. This leads to classifying certain distribution functions F such that the integral $\int_{-\infty}^{\infty} G(z-is)dF(s)$ has only real zeros as in Cardon [1] and Cardon-Nielsen [3].

A different observation is that $H_k(z,w)$, as a function of w, is a Fourier transform relative to a discrete measure. Theorem 1 says that, for fixed real z_0, $H_k(z_0,w)$ has only real zeros. A connection between the methods of Pólya and of Lee and Yang was noticed in a special case by Kac. See Kac's comments in [6, p.424-426]. In the process of studying the function $H_k(w) = H_k(0,w)$ we arrive at Theorem 1. The proof makes essential use of Pólya's Hilfssatz II suggesting the correctness of Kac's intuition.

2. Proof of Theorem 1

In this section we will prove Theorem 1. The task is to show that the zeros of $H_k(w)\,(=H_k(0,w))$ are real. To obtain information about $H_k(z,w)$ we consider an associated rational function of the variables $x_1,\ldots,x_k$ obtained by replacing e^{ib_jw} with x_j. This 'de-coupling' procedure was used by Lee and Yang [4]. Let

$$P_0(z;x) = G(z), \quad \text{and}$$
$$P_k(z;x) = P_{k-1}(z-ia_k;x)x_k^{-1} + P_{k-1}(z+ia_k;x)x_k \quad \text{for} \quad k>0,$$

where $x=(x_1,\ldots,x_k)$. Note that x is a vector of variables the number of which depends on the subscript in the expression P_k. For example,

$$P_1(z;x_1) = G(z-ia_1)x_1^{-1} + G(z+ia_1)x_1$$

and

$$\begin{aligned} P_2(z;x_1,x_2) = {} & G(z-ia_1-ia_2)x_1^{-1}x_2^{-1} + G(z+ia_1-ia_2)x_1x_2^{-1} \\ & + G(z-ia_1+ia_2)x_1^{-1}x_2 + G(z+ia_1+ia_2)x_1x_2. \end{aligned}$$

Note that P_k satisfies the fundamental relation

$$P_k(z;e^{b_1w},\ldots,e^{b_kw}) = H_k(z,w).$$

A nonrecursive definition of P_k is

$$P_k(z;x_1,\ldots,x_k) = \sum G(z \pm ia_1 \pm \cdots \pm ia_k) \prod_+ x_j \prod_- x_\ell^{-1}$$

where the summation is over all 2^k possible sign combinations. The expression $\prod_+ x_j$ means to take the product over those j corresponding to plus signs in the summation of the ia_m. Similarly, the expression $\prod_- x_\ell$ means to take the product over those ℓ corresponding to minus signs in the summation of the ia_m.

The most important step of the proof is Proposition 8 (below) in which we gain information about the zeros of $P_k(z;x_1,\ldots,x_k)$. Before getting to Proposition 8 we will deal with a few special cases and also prove two technical lemmas.

$P_k(z;x_1,\ldots,x_k)$ can degenerate into simpler cases for certain choices of a_m, b_m, or G. We will dispense with three trivial cases before proceeding with the main part of the proof. The first two trivial cases show why Theorems 1 and 2 assume that the real numbers $a_1,a_2,\ldots$ and $b_1,b_2,\ldots$ are nonzero:

Trivial Case 1. $P_n(z; x_1, \ldots, x_n)$ degenerates to a simpler case if $a_k = 0$ for some k. For example, if $a_n = 0$, then

$$H_n(z, w) = 2\cos(b_n w) H_{n-1}(z, w)$$

and

$$P_n(z; x_1, \ldots, x_n) = (x_n^{-1} + x_n) P_{n-1}(z; x_1, \ldots, x_{n-1}).$$

The problem of studying zeros of $H_n(z, w)$ and $P_n(z; x_1, \ldots, x_n)$ is reduced to the problem of studying the zeros of $H_{n-1}(z, w)$ and $P_{n-1}(z; x_1, \ldots, x_{n-1})$. Therefore we assume throughout this paper that $a_k > 0$ for all k.

Trivial Case 2. Another case to consider is when $b_k = 0$ for some k. For example, suppose $b_n = 0$. Define $\tilde{G}$ and $\tilde{H}$ by

$$\tilde{G}(z) = G(z - ia_n) + G(z + ia_n),$$

$$\tilde{H}_0(z, w) = \tilde{G}(z),$$

and

$$\tilde{H}_k(z, w) = \tilde{H}_{k-1}(z - ia_k, w) e^{-ib_k w} + \tilde{H}_{k-1}(z + ia_k, w) e^{ib_k w}$$

for $0 < k < n$. Then

$$H_n(z, w) = \tilde{H}_{n-1}(z, w).$$

This formula reduces the problem to the case of studying sums of exponential functions with fewer terms. To avoid this reduction we assume $b_k > 0$ for all k.

A third trivial case results from an especially simple choice for G:

Trivial Case 3. If $G(z)$ is an entire function of genus 0 or 1 that is real on the real axis and has no zeros at all, then $G(z)$ is of the form $ce^{\alpha z}$ where c is a nonzero real number and α is real. Then

$$H_n(z, w) = ce^{\alpha z} \prod_{k=1}^{n} 2\cos(b_k w + \alpha a_k)$$

and

$$P_n(z; x_1, \ldots, x_n) = ce^{\alpha z} \prod_{k=1}^{n} (x_k e^{i\alpha a_k} + x_k^{-1} e^{-i\alpha a_k}).$$

Then, $H_n(0, w)$ has only real zeros. This simple case is included in the statement of Theorems 1 and 2; however, it is excluded from the statement of Proposition 8 (below) because the proposition assumes that $G(z)$ has at least $n \geq 1$ real zeros.

The following two technical lemmas are required later and describe the number of zeros of $G(z - ia)e^{-ib} + G(z + ia)e^{ib}$.

Lemma 6 *Let $G(z)$ be an entire function of genus 0 or 1 that is real for real z, has only real zeros, and has at least $n \geq 1$ zeros (counting multiplicity). Let $a > 0$ and let b be real. Then*

$$G(z-ia)e^{-ib} + G(z+ia)e^{ib}$$

is also a function of genus 0 or 1 that is real on the real axis, has only real zeros, and has at least $n-1$ zeros.

Proof. Assume G has $n \geq 1$ zeros. Then $G(z)$ is of the form

$$G(z) = e^{\alpha z}(c_n z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0)$$

where $c_n \neq 0$. Let

$$H(z) = G(z-ia)e^{-ib} + G(z+ia)e^{ib}.$$

Then by expanding and collecting powers of z we obtain

$$H(z) = e^{\alpha z}(d_n z^n + d_{n-1}z^{n-1} + \cdots + d_1 z + d_0)$$

where

$$d_n = 2c_n \cos(b + \alpha a) \quad \text{and} \quad d_{n-1} = 2c_{n-1}\cos(b+\alpha a) - 2anc_n \sin(b+\alpha a).$$

If $\cos(b+\alpha a) \neq 0$, then $d_n \neq 0$ and $H(z)$ has n zeros counting multiplicities. If $\cos(b+\alpha a) = 0$, then $d_n = 0$ but $d_{n-1} \neq 0$ in which case $H(z)$ has $n-1$ zeros counting multiplicities. □

Lemma 7 *Let $G(z)$ be an entire function of genus 0 or 1 that is real for real z, has only real zeros, and has infinitely many zeros. Let $a > 0$ and let b be real. Then*

$$G(z-ia)e^{-ib} + G(z+ia)e^{ib}$$

is also a function of genus 0 or 1 that is real on the real axis, has only real zeros, and has infinitely many zeros.

Proof. We begin with a simple observation about real entire functions of genus 0 or 1. Let ϕ be such a function. Then ϕ may be represented as

$$\phi(z) = cz^m e^{\alpha z} \prod_k (1 - z/\alpha_k)e^{z/\alpha_k}$$

where c, α, and α_k are real and m is a nonnegative integer. For any real T,

$$|\phi(iT)|^2 = T^{2m} \prod_k (1 + T^2/\alpha_k^2).$$

Thus $\phi(z)$ has infinitely many zeros if and only if $|\phi(iT)|^2$ grows more rapidly than any power of T.

Now let

$$H(z) = G(z-ia)e^{-ib} + G(z+ia)e^{ib}.$$

By making a change of variable, if necessary, there is no loss of generality in assuming that G is of the form

$$G(z) = e^{\alpha z} \prod_{k=1}^{\infty} (1 - z/\alpha_k) e^{z/\alpha_k}$$

where the α_k are the real zeros of G. Let

$$g(z) = \prod_{k=1}^{\infty} (1 - z^2/\alpha_k^2).$$

For real T, $g(iT) = |G(iT)|^2$. Since $g(z)$ satisfies the conditions of Proposition 3, the derivative $g'(z) = \lim_{h\to 0} \frac{g(z+ih)-g(z-ih)}{2ih}$ also satisfies the conditions of Proposition 3. Thus, $g'(z)$ is of the form

$$g'(z) = z \prod_{k=1}^{\infty} (1 - z^2/\beta_k^2).$$

where the β_k are real and $\sum \beta_k^{-2} < \infty$.

By the observation at the beginning of the proof, since $G(z)$ has infinitely many zeros, $g(iT) = |G(iT)|^2$ grows more rapidly than any power of T. Similarly, for fixed real a, both $g(iT + ia) = |G(iT + ia)|^2$ and $g(iT - ia) = |G(iT - ia)|^2$ grow more rapidly than any power of T. By showing that the difference

$$g(iT + ia) - g(iT - ia) = |G(iT + ia)|^2 - |G(iT - ia)|^2$$

grows more rapidly than any power of T we may conclude that $|H(iT)|^2$ also grows rapidly and hence that $H(z)$ has infinitely many zeros.

By the mean value theorem of calculus there exists a real number a_T depending on T in the interval $(-a, a)$ such that

$$g(iT + ia) - g(iT - ia) = 2aig'(iT + ia_T).$$

Since $g(iT)$ and $ig'(iT)$ are increasing functions of positive T,

$$g(iT + ia) - g(iT - ia) \geq 2aig'(iT - ia)$$

for all $T \geq a$. But the right hand side grows more rapidly than any power of T. Therefore, $|G(iT + ia)|^2$ grows sufficiently more rapidly than $|G(iT - ia)|^2$ to conclude that $|H(iT)|^2 = |G(iT + ia)e^{ib} + G(iT - ia)e^{-ib}|^2$ grows more rapidly than any polynomial as T becomes large. Thus, $H(z)$ has infinitely many zeros. $\square$

The following proposition is similar to the result of Lee and Yang mentioned in §1 (see Proposition 4).

Proposition 8 *Let $G(z)$ be an entire function of genus 0 or 1 that has only real zeros, has at least $n \geq 1$ real zeros (counting multiplicity), and is real for real z.*

(i) Suppose $P_n(iA; x_1, \ldots, x_n) = 0$ *and* $A > 0$*,* $a_1 > 0, \ldots, a_n > 0$*,* $|x_1| \geq 1$*,* $\ldots$*,* $|x_{n-1}| \geq 1$*. Then* $|x_n| < 1$*.*
(ii) Suppose $P_n(-iA; x_1, \ldots, x_n) = 0$ *and* $A > 0$*,* $a_1 > 0, \ldots, a_n > 0$*,* $|x_1| \leq 1$*,* $\ldots$*,* $|x_{n-1}| \leq 1$*. Then* $|x_n| > 1$*.*

Proof. We will prove part (i) by induction on n. The proof of part (ii) is identical except for reversing some inequalities. In the case $n = 1$, suppose

$$0 = P_1(iA; x_1) = G(iA - ia_1)x_1^{-1} + G(iA + ia_1)x_1.$$

The hypotheses on G imply that for real c, d with $0 \leq c < d$, $|G(ic)| < |G(id)|$. Then $|A - a_1| < |A + a_1|$ implies $|x_1| = \left|\frac{G(iA-ia_1)}{G(iA+ia_1)}\right|^{1/2} < 1$.

Now assume $n \geq 2$ and that the theorem holds for P_k with $1 \leq k < n$. If, by way of contradiction, the theorem is false for P_n, there exists a solution $x_1, \ldots, x_n$ of the equation $0 = P_n(iA; x_1, \ldots, x_n)$ such that $|x_k| \geq 1$ for $1 \leq k \leq n$. We will show that this leads to the existence of another solution $w_1, \ldots, w_n$ such that

$$|w_1| = 1, \ldots, |w_{n-1}| = 1, |w_n| \geq 1.$$

From the definition of P_n we have

$$\begin{aligned} 0 &= P_n(iA; x_1, \ldots, x_n) \\ &= P_{n-2}(iA - ia_{n-1} - ia_n; x)x_{n-1}^{-1}x_n^{-1} + P_{n-2}(iA - ia_{n-1} + ia_n; x)x_{n-1}^{-1}x_n \\ &\quad + P_{n-2}(iA + ia_{n-1} - ia_n; x)x_{n-1}x_n^{-1} + P_{n-2}(iA + ia_{n-1} + ia_n; x)x_{n-1}x_n. \end{aligned}$$

By the induction hypothesis $P_{n-2}(iA + ia_{n-1} + ia_n; x_1, \ldots, x_{n-2}) \neq 0$. The last equation shows that x_{n-1}^2 and x_n^2 are related through the fractional linear transformation:

$$x_{n-1}^2 = -\frac{P_{n-2}(iA - ia_{n-1} - ia_n; x) + P_{n-2}(iA - ia_{n-1} + ia_n; x)x_n^2}{P_{n-2}(iA + ia_{n-1} - ia_n; x) + P_{n-2}(iA + ia_{n-1} + ia_n; x)x_n^2}.$$

As x_n^2 tends to ∞, x_{n-1}^2 tends to the value ${x'}_{n-1}^2$ where

$${x'}_{n-1}^2 = -\frac{P_{n-2}(iA - ia_{n-1} + ia_n; x)}{P_{n-2}(iA + ia_{n-1} + ia_n; x)}.$$

For the value ${x'}_{n-1}^2$ we have

$$\begin{aligned} 0 &= P_{n-2}(iA - ia_{n-1} + ia_n; x) + P_{n-2}(iA + ia_{n-1} + ia_n; x){x'}_{n-1}^2 \\ 0 &= x'_{n-1}P_{n-1}(iA + ia_n; x_1, \ldots, x_{n-2}, x'_{n-1}). \end{aligned}$$

Since $|x_1| \geq 1, \ldots, |x_{n-2}| \geq 1$, the induction hypothesis implies $|x'_{n-1}| < 1$. Therefore, by continuity, there exists a solution $x_1, \ldots, x_{n-2}, w_{n-1}, \tilde{x}_n$ such that

$$|x_1| \geq 1, \ldots, |x_{n-2}| \geq 1, |w_{n-1}| = 1, |\tilde{x}_n| \geq |x_n| \geq 1.$$

Repeating this argument with the indices $1, \ldots, n-2$ in place of $n-1$ results in a solution $w_1, \ldots, w_n$ such that

$$|w_1| = 1, \ldots, |w_{n-1}| = 1, |w_n| \geq |x_n| \geq 1.$$

Then

$$\begin{aligned} 0 &= P_n(iA; w_1, \ldots, w_n) \\ 0 &= P_{n-1}(iA - ia_n; w_1, \ldots, w_{n-1})w_n^{-1} + P_{n-1}(iA + ia_n; w_1, \ldots, w_{n-1})w_n. \end{aligned}$$

Let $f_k(z)$ for $1 \leq k \leq n-1$ be defined recursively by

$$\begin{aligned} f_1(z) &= G(z - ia_1)w_1^{-1} + G(z + ia_1)w_1, \\ f_k(z) &= f_{k-1}(z - ia_k)w_k^{-1} + f_{k-1}(z + ia_k)w_k. \end{aligned}$$

Note that

$$\begin{aligned} f_{n-1}(iA - ia_n) &= P_{n-1}(iA - ia_n; w_1, \ldots, w_{n-1}), \quad \text{and} \\ f_{n-1}(iA + ia_n) &= P_{n-1}(iA + ia_n; w_1, \ldots, w_{n-1}). \end{aligned}$$

By Lemmas 6 and 7, $f_k(z)$ for $1 \leq k \leq n-1$ is a function of genus 0 or 1 that is real for real z and has only real zeros and has at least one real zero. Then $|A - a_n| < |A + a_n|$ implies $|f_{n-1}(iA - ia_n)| < |f_{n-1}(iA + ia_n)|$ giving

$$|w_n|^2 = \left| \frac{f_{n-1}(iA - ia_n)}{f_{n-1}(iA + ia_n)} \right| < 1$$

which contradicts the fact that $|w_n| \geq |x_n| \geq 1$. Therefore, the assumption that $|x_n| \geq 1$ is false. Hence, $|x_n| < 1$. □

Corollary 9 *Let $G(z)$ be an entire function of genus 0 or 1 that has only real zeros, has at least $n \geq 1$ real zeros (counting multiplicity), and is real for real z. Suppose $A > 0$, $a_1 > 0, \ldots, a_n > 0$, $b_1 > 0, \ldots, b_n > 0$. Let $\Im(w)$ denote the imaginary part of w.*

(i) If $H_n(iA, w) = 0$, then $\Im(w) > 0$.
(ii) If $H_n(-iA, w) = 0$, then $\Im(w) < 0$.
(iii) If $H_n(0, w) = 0$, then w is real.

Proof. Set $x_k = e^{ib_k w}$. By Proposition 8 if $|x_k| = |e^{ib_k w}| \geq 1$ for each k, then $H_n(iA, w) \neq 0$. So, $H_n(iA, w) = 0$ implies $|x_k| = |e^{ib_k w}| < 1$ for some k which means that $\Im(w) > 0$. This proves (i). The proof of part (ii) is similar.

By part (i), since $\lim_{A \to 0^+} H_n(iA, w) = H_n(0, w)$ is uniform on compact sets, the zeros of $H_n(0, w)$ must have $\Im(w) \geq 0$. Similarly, $\lim_{A \to 0^+} H_n(-iA, w) = H_n(0, w)$ so that the roots satisfy $\Im(w) \leq 0$. But then $\Im(w) = 0$ and we conclude that $H_n(0, w)$ has only real roots. This proves (iii). □

Recalling that $H_n(w) = H_n(0, w)$ and by forming the associated polynomial $P_n(t)$ we obtain the following special cases of Theorems 1 and 2 as immediate consequences of Corollary 9.

Proposition 10 *Let $G(z)$ be an entire function of genus* 0 *or* 1 *that has only real zeros, has at least $n \geq 1$ real zeros* (*counting multiplicity*)*, and is real for real z. Then*

(i) all the zeros of the exponential sum $H_n(w)$ are real, and
(ii) all the zeros of the polynomial $P_n(t)$ lie on the unit circle in the complex plane.

Now we wish to remove the artificial restriction that $G(z)$ must have at least $n \geq 1$ zeros in the previous proposition. The case not covered is when $G(z)$ is of the form

$$G(z) = cz^q e^{\alpha z} \prod_{m=1}^{k} (1 - z/\alpha_m)$$

where q is a nonnegative integer, c and α are real, the α_m are the nonzero real zeros of G, and $0 \leq k < n$. For positive N let

$$G_N(z) = (1 - z/N)^{n-k} \cdot cz^q e^{\alpha z} \prod_{m=1}^{k} (1 - z/\alpha_m) .$$

Then $G_N(z)$ has n real roots. Let $H_{N,n}(z)$ be the exponential sum formed using $G_N(z)$. That is,

$$H_{N,n}(w) = \sum G_N(\pm ia_1 \pm \cdots \pm ia_n) e^{iw(\pm b_1 \pm \cdots \pm b_n)}$$

where the summation is over all 2^n possible plus and minus sign combinations, the same sign combination being used in both the argument of G and in the exponent. By Proposition 10, $H_{N,n}(w)$ has only real roots.

Since $\lim_{N\to\infty} H_{N,n}(w) = H_n(w)$ and the convergence is uniform on compact sets, the zeros of $H_n(w)$ are real because they are limit points of the zeros of the $H_{N,n}(w)$. This completes the proof of Theorem 1.

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