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CONVOLUTION OPERATORS AND ZEROS OF ENTIRE FUNCTIONS

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ABSTRACT. Let $G(z)$ be a real entire function of order less than 2 with only real zeros. Then we classify certain distributions functions F such that the convolution $(G * dF)(z) = \int_{-\infty}^{\infty} G(z - is)\, dF(s)$ has only real zeros.

1. INTRODUCTION

The main result of this paper is the following theorem:

Theorem 1. *Suppose G is an entire function of order < 2 that is real on the real axis and has only real zeros. Let $\{a_i\}$ be a nonincreasing sequence of positive real numbers, let $\{X_i\}$ be the sequence of independent random variables such that X_i takes values ± 1 with equal probability, and let F_n be the distribution function of the normalized sum $Y_n = (a_1 X_1 + \cdots + a_n X_n)/s_n$ where $s_n^2 = a_1^2 + \cdots + a_n^2$. The functions F_n converge pointwise to a continuous distribution $F = \lim_{n\to\infty} F_n$. Define H by the integral*

$$H(z) = (G * dF)(z) = \int_{-\infty}^{\infty} G(z - is)\, dF(s).$$

Then H is an entire function of order < 2 that is real on the real axis. If H is not identically zero, then H has only real zeros.

The organization of this paper is as follows: This section is an introduction to the problem under investigation. Section 2 contains the proof of Theorem 1 as a sequence of lemmas. Finally, in Section 3 we give a number of examples and state several questions for further study.

Theorem 1 generalizes a very interesting observation by Pólya about the zeros of certain entire functions:

Theorem 2 (Pólya [18], Hilfssatz II)**.** *Let a be a positive constant and let $G(z)$ be an entire function of genus 0 or 1 that for real z takes real values, has at least one real zero, and has only real zeros. Then the function $G(z + ia) + G(z - ia)$ has only real zeros.*

For the benefit of the reader and to illustrate Pólya's elegant idea, we include his short proof of Theorem 2.

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Proof. By hypothesis G has a Weierstrass product of the form

$$G(z) = cz^q e^{\alpha z} \prod (1 - z/\alpha_n)\, e^{z/\alpha_n}$$

where q is a nonnegative integer, c and α are real, and the α_n are the nonzero real zeros of G. Suppose $z = x + iy$ is a zero of $G(z + ia) + G(z - ia)$. Then

$$|G(z + ia)| = |G(z - ia)|$$

and

$$1 = \left|\frac{G(z - ia)}{G(z + ia)}\right|^2 = \left(\frac{x^2 + (y - a)^2}{x^2 + (y + a)^2}\right)^q \prod \frac{(x - \alpha_n)^2 + (y - a)^2}{(x - \alpha_n)^2 + (y + a)^2}.$$

If $y > 0$ then the right hand side of the last expression is < 1. If $y < 0$ then the right hand side of the last expression is > 1. Both of these cases are impossible. Hence, $y = 0$ and $G(z + ia) + G(z - ia)$ has only real zeros. $\square$

Pólya's application of Theorem 2 was to study the Riemann zeta function $\zeta(s)$. Let

$$\xi(s) = \tfrac{1}{2}s(s - 1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

It is well known that $\xi(s)$ is an entire function satisfying $\xi(s) = \xi(1 - s)$ and all of its zeros lie in the 'critical strip' $0 < \Re(s) < 1$. The Riemann Hypothesis predicts that $\Re(s) = 1/2$ for all zeros of $\xi(s)$. By considering the main term resulting from the Fourier integral representation of $\xi(1/2 + it)$ Pólya approximated $\xi(s)$ with a 'fake' zeta function $\xi^*(s)$:

$$\xi(\tfrac{1}{2} + it) \approx \xi^*(\tfrac{1}{2} + it) = 4\pi^2\big(K_{9/4+it/2}(2\pi) + K_{9/4-it/2}(2\pi)\big) \tag{1}$$

where $K_z(\alpha)$ is the K-Bessel function defined by

$$K_z(\alpha) \equiv \int_0^\infty e^{-\alpha\cosh w}\cosh(zw)\, dw.$$

He proved that $K_{it/2}(2\pi)$ has only real zeros. Then applying Theorem 2 with the shift $a = 9/2$ he showed that the sum of Bessel functions has zeros only when t is real. Thus, each zero of $\xi^*(s)$ is on the 'critical line' $\Re(s) = 1/2$. Furthermore, he showed that asymptotically $\xi(s)$ and $\xi^*(s)$ have the same number of zeros in the critical strip.

Later Pólya [19] refined his approximation in (1) to include several more terms. In a very interesting paper in 1990 Hejhal [13] vastly generalized Pólya's line of reasoning. In this paper we will not pursue Pólya's or Hejhal's investigations of the zeta function. Rather, we will focus on generalizing Theorem 2. De Bruijn has also generalized Theorem 2, but in a way different from Theorem 1. (See Theorem 3 of [3].)

The fact that the function $\xi(\frac{1}{2} + it)$ may be represented as a Fourier transform has motivated interest in classifying the functions K such that

$$G(z) = \int_{-\infty}^\infty K(t)e^{izt}\, dt$$

is an entire function with only real zeros. Several authors have dealt with this topic or more generally with functions in the Laguerre-Pólya class in recent years. (See, for example, the papers by Craven, Csordas, Smith, Varga, and Vincze : [4], [5], [6],[7],[8].)

It is interesting to apply Theorem 1 when G can be represented as a Fourier transform. We obtain the following:

Theorem 3. *Suppose G is an entire function of order < 2 that is real on the real axis and has only real zeros. Also suppose that G may be represented as*

$$G(z) = \int_{-\infty}^{\infty} K(t)e^{itz}\,dt$$

where $K(t)$ and $|K(t)|$ are integrable real functions of t such that $K(t) = O(e^{-|t|^{2+\epsilon}})$ for any $\epsilon > 0$. Let F be the distribution function in Theorem 1. Define H by

$$H(z) = \int_{-\infty}^{\infty} K(t)L(t)e^{izt}dt$$

where $L(t) = \int_{-\infty}^{\infty} \cosh(ts)\,dF(s)$. If H is not identically zero, then all of the zeros of H are real.

Proof.

$$\begin{aligned}
H(z) &= \int_{-\infty}^{\infty} K(t)\left(\int_{-\infty}^{\infty} \cosh(ts)\,dF(s)\right) e^{izt}\,dt \\
&= \int_{-\infty}^{\infty} K(t)\left(\int_{-\infty}^{\infty} e^{ts}\,dF(s)\right) e^{izt}\,dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} K(t)e^{it(z-is)}\,dt\right) dF(s) \\
&= \int_{-\infty}^{\infty} G(z-is)\,dF(s).
\end{aligned}$$

By Theorem 1, if H is not the zero function then H has only real zeros. □

Notice that in certain cases it is not difficult to explicitly evaluate the integral for L. For example, if F is the normal distribution with variance 1 we obtain $L(t) = e^{t^2/2}$. If F is not the normal distribution its tail will not be as heavy as that of the normal distribution (see Lemma 3 below) in which case L may grow more slowly than $e^{t^2/2}$.

2. Proof of Theorem 1

The proof of Theorem 1 will be presented as a sequence of lemmas. But first we need to establish some notation.

The Laguerre-Pólya class $\mathcal{LP}$ of functions consists of the entire functions having only real zeros with a Weierstrass factorization of the form

$$az^q e^{\alpha z - \beta z^2} \prod (1 - z/\alpha_n)e^{z/\alpha_n}$$

where a, α, β are real, $\beta \geq 0$, q is a nonnegative integer, and the α_n are nonzero real numbers such that $\sum_{n=1}^{\infty} \alpha_n^{-2} < \infty$. We shall be most interested in the subset $\mathcal{LP}^*$ of the Laguerre-Pólya class consisting of all elements of $\mathcal{LP}$ of order < 2. Thus, β is necessarily 0 for functions in $\mathcal{LP}^*$.

The distribution function T for a random variable Y is $T(x) = \Pr(Y \leq x)$. We will consider the following types of random variables and their distribution functions: Let $\{a_i\}$ be a nonincreasing sequence of positive real numbers. Let $\{X_i\}$ be a sequence of independent random variables such that X_i takes values ± 1 with equal probability. Let Y_n be the sum

$$Y_n = \frac{a_1 X_1 + \cdots + a_n X_n}{s_n}$$

where $s_n^2 = a_1^2 + \cdots + a_n^2$. F_n will denote the distribution function of Y_n, and F will denote the limit $F = \lim_{n\to\infty} F_n$. In Theorem 1 the distribution F has variance 1. However, F could be rescaled to have any other positive value for its variance. The following lemma describes this F:

Lemma 1. *The sequence F_n converges pointwise to a continuous distribution F. If the sequence s_n is unbounded, F is the normal distribution. If the sequence s_n is bounded, F is not the normal distribution.*

Proof. There are two basic types of limiting distributions F that can occur in this setting. First, the variances s_n^2 of the sums $a_1X_1 + \cdots + a_nX_n$ could be unbounded for increasing n. This condition causes the contribution due to individual summands of $Y_n = (a_1X_1 + \cdots + a_nX_n)/s_n$ to be 'negligible' as n becomes large. In this case, the only possible limiting distribution F is the normal distribution. This is a result of the Lévy-Khinchin Representation Theorem (see [12, Thm. 24, p. 310]).

Second, the sum $\sigma^2 = \sum_{i=1}^\infty a_i^2$ can be finite. Kolmogorov's Three Series Theorem (see [12, p. 203] or [15, p. 237]) guarantees that the distribution functions F_n converge to a distribution function F almost surely. We must show that $\lim_{n\to\infty} F_n(x) = F(x)$ exists for all x and that F is continuous.

In this case, F is not the normal distribution. This can be seen by considering the characteristic function f of F which is $f(u) = \int e^{iux} dF(x) = \prod_{i=1}^\infty \cos(a_i u/\sigma)$. Since f vanishes for certain values of u, the Lévy-Khinchin Representation Theorem (see [15, p. 301] or [12, p. 299]) shows that F is not infinitely divisible and therefore is not a normal distribution.

Let $b_1, b_2, \ldots$ be a subsequence of $a_1, a_2, \ldots$ such that $b_{i+1} < b_i/3$ for all i. Let Z_i be the corresponding subsequence of the X_i. Suppose that the distribution function S_n for the random variable

$$W_n = b_nZ_n + b_{n+1}Z_{n+1} + \cdots$$

has maximal atom size M_n. Because the sequence $\{b_i\}$ decreases sufficiently rapidly, $b_n + W_{n+1}$ and $-b_n + W_{n+1}$ have no values in common. Since

$$S_n(x) = 2^{-1}\big(S_{n+1}(x + b_n) + S_{n+1}(x - b_n)\big),$$

it follows that $M_n = M_{n+1}/2$. But M_n is bounded by 1 for all n which implies that $M_1 = 0$. This shows that S_1 is a continuous distribution function. Now we decompose $\sum_i a_iX_i$ as the sum

$$\sum a_iX_i = W_1 + \sum{}' a_iX_i$$

where the prime on the summation symbol means to exclude the terms corresponding to the subsequence $\{b_i\}$. Since the convolution of the non-atomic probability distribution S_1 with the probability distribution for $\sum' a_iX_i$ is non-atomic it follows that $\sum a_iX_i$ has a continuous distribution function. Therefore F is continuous. Since the functions F_n are increasing functions that converge almost everywhere to F, $\lim_{n\to\infty} F_n(x) = F(x)$ exists pointwise. □

We will now establishing the fact that Pólya's result may be applied repeatedly.

Lemma 2. *Suppose $G \in \mathcal{LP}^*$.*

(a) *Let $H(z) = \frac{1}{2}\left(G(z - ia) + G(z + ia)\right)$ for $a > 0$. Then $H \in \mathcal{LP}^*$.*

(b) *Let* $a_1, \dots, a_n$ *be positive real numbers. Then the function*

$$2^{-n} \sum G(z \pm ia_1 \pm \cdots \pm ia_n)$$

obtained by summing over all possible combinations of plus and minus signs is also contained in $\mathcal{LP}^*$.

(c) *The convolution* H_n *of* G *with the measure* dF_n,

$$H_n(z) = (G * dF_n)(z) = \int_{-\infty}^{\infty} G(z - is)\, dF_n(s),$$

is in $\mathcal{LP}^*$.

Proof. It is clear that $H(z)$ is real for real z and that the order of H is < 2. Theorem 2 guarantees that if G has only real zeros and at least one real zero then H has only real zeros. So in this case, $H \in \mathcal{LP}^*$. The only other cases to consider are when $G(z) = c$ or $ce^{\alpha z}$ where c and α are real. But then $H(z) = c$ or $\cos(\alpha a)ce^{\alpha z}$. In either case $H \in \mathcal{LP}^*$. This proves part (a).

For (b) apply part (a) several times. For part (c) replace each a_i by the normalized values a_i/s_n where $s_n^2 = a_1^2 + \cdots + a_n^2$ and use the notation of a Riemann-Stieltjes integral. □

In Lemmas 4 and 5 we will show that the integrals $\int_{-\infty}^{\infty} G(z-is)\, dF_n(s)$ converge uniformly to $\int_{-\infty}^{\infty} G(z-is)\, dF(s)$ for z in compact sets. The proof of Lemma 4 will require the following 1994 result of Pinelis which is an improvement of a conjecture by Eaton [9]:

Lemma 3 (Pinelis [17], Corollary 2.6)**.** *Let* X_i *be independent random variables taking values* ± 1 *with equal probability. Let* $s_n^2 = a_1^2 + \cdots + a_n^2$ *and let*

$$Y_n = \frac{a_1 X_1 + \cdots + a_n X_n}{s_n}.$$

Then

$$\Pr(|Y_n| > u) < 2c(1 - \Phi(u))$$

where $c = 2e^3/9$, $\Phi(u) = \int_{-\infty}^{u} \phi(t)\, dt$, *and* $\phi(t) = (2\pi)^{-1/2} e^{-t^2/2}$.

Lemma 4. *Write* $G_z(s) = G(z - is)$. *Let* $\epsilon > 0$ *be given, suppose* $G \in \mathcal{LP}^*$, *and let* K *be a compact subset of* $\mathbb{C}$. *Then there is a positive number* A (*depending on* ϵ *and* K) *such that*

$$\int_{|s|>A} |G_z(s)|\, dF_n(s) < \epsilon$$

for all n *and all* $z \in K$.

Proof. Let λ denote the order of G. Choose δ with $\lambda < \delta < 2$. Then choose $A > 0$ large enough so that $|G_z(s)| < e^{|s|^\delta}$ for all $z \in K$ and $|s| > A$. Such an A exists as follows: Choose δ' with $\lambda < \delta' < \delta$. Because the order of G is less than δ' there exists an R such that $|G(s)| < e^{|s|^{\delta'}}$ for $|s| > R$. Choose κ large enough so that $|z| < \kappa$ for all $z \in K$. If $|s| > \kappa + R$,

$$|G_z(s)| = |G(z - is)| < e^{|z-is|^{\delta'}} < e^{(\kappa+|s|)^{\delta'}}.$$

Then there is an $A > \kappa + R$ such that if $|s| > A$ then $e^{(\kappa+|s|)^{\delta'}} < e^{|s|^\delta}$.

Then

$$\int_{A<|s|<B} |G_z(s)|dF_n(s) < 2\int_A^B e^{s^\delta} dF_n(s).$$

After integration by parts the right hand side becomes

$$\begin{aligned}&2e^{B^\delta}(F_n(B)-1) - 2e^{A^\delta}(F_n(A)-1) - 2\int_A^B (F_n(s)-1)d(e^{s^\delta})\\ &< 2e^{A^\delta}(1-F_n(A)) + 2\int_A^B (1-F_n(s))\delta s^{\delta-1}e^{s^\delta}ds.\end{aligned}$$

According to Lemma 3 for $s \geq A$ and if A is sufficiently large, then

$$1 - F_n(s) \leq \beta \int_s^\infty e^{-t^2/2}dt < \frac{e^{-s^2/2}}{2}$$

where $\beta = 2e^3/(9\sqrt{2\pi})$. This gives

$$\int_{A<|s|<B} |G_z(s)|dF_n(s) < e^{A^\delta - A^2/2} + \int_A^B \delta s^{\delta-1} e^{s^\delta - s^2/2}ds$$

and

$$\int_{A<|s|} |G_z(s)|dF_n(s) < e^{A^\delta - A^2/2} + \int_A^\infty \delta s^{\delta-1} e^{s^\delta - s^2/2}ds.$$

For sufficiently large A the last integral is bounded above by $e^{A^\delta - A^2/2}$. So,

$$\int_{A<|s|<B} |G_z(s)|dF_n(s) < 2e^{A^\delta - A^2/2}.$$

The right hand side of the last inequality can be made arbitrarily small for sufficiently large A. Therefore, we obtain $\int_{|s|>A} |G_z(s)|dF_n(s) < \epsilon$ as desired. □

Lemma 5. *Let K be a compact subset of $\mathbb{C}$. Then*

$$\int_{-A}^A G_z(s)\, dF_n(s) \to \int_{-A}^A G_z(s)\, dF(s)$$

uniformly as $n \to \infty$ for $z \in K$.

Proof. By the Helly-Bray Theorem (see [15, p. 182] or [10, p. 298]) it is immediate that convergence occurs pointwise. We must, however, verify uniform convergence for $z \in K$.

Let $\epsilon > 0$ be given. Choose κ such that $\kappa > |G_z(s)| = |G(z - is)|$ and $\kappa > |G'(z - is)|$ for all $z \in K$ and $s \in [-A, A]$. Integration by parts yields

$$\int_{-A}^A G_z(s)\, dF(s) = G_z(A)F(A) - G_z(-A)F(-A) + i\int_{-A}^A F(s)G'(z - is)\, ds.$$

Then

$$\begin{aligned}
&\left|\int_{-A}^{A} G_z(s)\,dF(s) - \int_{-A}^{A} G_z(s)\,dF_n(s)\right| \\
&\le |G_z(A)|\,|F(A)-F_n(A)| + |G_z(-A)|\,|F(-A)-F_n(-A)| \\
&\qquad + \int_{-A}^{A} |F(s)-F_n(s)|\,|G'(z-is)|\,ds \\
&\le \kappa|F(A)-F_n(A)| + \kappa|F(-A)-F_n(-A)| + \kappa 2A \max_{s\in[-A,A]} |F(s)-F_n(s)|
\end{aligned}$$

Since the functions F_n and F are distributions functions such that F_n converges to the continuous distribution F pointwise on $[-A, A]$, F_n converges to F uniformly on $[-A, A]$. Thus for sufficiently large n and all $s \in [-A, A]$

$$|F_n(s) - F(s)| < \min\left(\frac{\epsilon}{6A\kappa}, \frac{\epsilon}{3\kappa}\right).$$

Therefore, for all sufficiently large n and all $z \in K$,

$$\left|\int_{-A}^{A} G_z(s)\,dF(s) - \int_{-A}^{A} G_z(s)\,dF_n(s)\right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This shows that the convergence $\int_{-A}^{A} G_z(s)\,dF_n(s) \to \int_{-A}^{A} G_z(s)\,dF(s)$ is uniform as claimed. ☐

Lemma 6. *Suppose $G \in \mathcal{LP}^*$. Then $H(z) = \int_{-\infty}^{\infty} G(z-is)\,dF(s)$ is an entire function that is real for real z and if it does not vanish identically then it has only real zeros.*

Proof. Since $H_n(z)$ is real for real z, $H(z)$ is real for real z.

Let $H_n(z) = \int_{-\infty}^{\infty} G(z-is)\,dF_n(s)$. Then $H_n(z)$ converges uniformly to $H(z)$ on compact sets by Lemmas 4 and 5. By Hurwitz's Theorem (see [1, Thm. 2, p. 178]), if H is not identically zero, then $H(z)$ is nonzero for nonreal z and the zeros of $H(z)$ are limit points of the zeros of $H_n(z)$ which are real. ☐

Lemma 7. *Suppose $G \in \mathcal{LP}^*$. The order of $H(z) = \int_{-\infty}^{\infty} G(z-is)\,dF(s)$ is $\le \lambda$ where λ is the order of G.*

Proof. Choose δ with $\lambda < \delta < 2$ where λ is the order of G. Then there is an R such that $|G(z)| < e^{|z|^\delta}$ for $|z| > R$. Setting $M = \max_{|z|\le R}\{1, |G(z)|\}$ we obtain $|G(z)| \le Me^{|z|^\delta}$ for all z. Also recall that for any complex numbers α and β and positive δ we have the inequality $|\alpha+\beta|^\delta \le \gamma(|\alpha|^\delta + |\beta|^\delta)$ where $\gamma = \max\{1, 2^{\delta-1}\}$ (see [16, Ex. 17, p. 73]). Then

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} G(z-is)\,dF(s)\right| &\le \int_{-\infty}^{\infty} |G(z-is)|\,dF(s) \le \int_{-\infty}^{\infty} Me^{|z-is|^\delta}\,dF(s) \\
&\le \int_{-\infty}^{\infty} Me^{(|z|+|s|)^\delta}\,dF(s) \le \int_{-\infty}^{\infty} Me^{\gamma|z|^\delta+\gamma|s|^\delta}\,dF(s)
\end{aligned}$$

By Lemma 3 we may conclude that the integral $\int_{-\infty}^{\infty} e^{\gamma|s|^\delta}\,dF(s)$ is finite. Hence, $|H(z)|$ is bounded by a constant times $e^{\gamma|z|^\delta}$. This implies that the order of H does not exceed λ which is less than 2. ☐

This completes the proof of Theorem 1.

3. Examples and Questions

1. Let $G(z) = z^n$ and $a > 0$. Then $H(z) = \frac{1}{2}(G(z+ia) + G(z-ia))$ has n simple real zeros which become more widely spaced as a is increased. The 'shifted sum operator' of Theorem 2 has the effect of causing the zeros to repel.

Similarly, suppose $G(z) = z^n G_0(z)$ where $G_0 \in \mathcal{LP}^*$ and G_0 has no zeros at $z = 0$. If z is sufficiently close to zero then $G_0(z)$ behaves as a nonzero constant. Thus for sufficiently small a, the zero of multiplicity n of $G(z)$ splits into n simple zeros of $H(z)$. Again, it appears that the shifted sum causes zeros to repel.

It would be interesting to determine what additional hypotheses (if any) are needed in Theorem 1 to guarantee that if $G \in \mathcal{LP}^*$ then the zeros of $G * dF$ are simple.

2. The sine and cosine functions are eigenfunctions of the convolution with dF for any distribution F discussed in this paper. This is because the 'shifted sum operator' has the effect

$$\frac{1}{2}(\sin(z+ia) + \sin(z-ia)) = \cosh(a)\sin(z)$$
$$\frac{1}{2}(\cos(z+ia) + \cos(z-ia)) = \cosh(a)\cos(z).$$

For the distribution F, if $G(z)$ is either $\sin(z)$ or $\cos(z)$, we obtain

$$(G * dF)(z) = \lambda\, G(z) \quad \text{where} \quad \lambda = \int_{-\infty}^{\infty} \cosh(s)\, dF(s).$$

The results for the sine and cosine functions are consistent with the heuristic given in the first example because if evenly spaced zeros repel each other then the result will be a function with evenly spaced zeros.

3. In Theorem 1 let $a_i = 1$ for all i and normalize the partial sums to have variance $1/\sqrt{2}$ instead of 1. Then the distribution functions for the random variables $Y_n = \frac{1}{\sqrt{2n}}(X_1 + \cdots + X_n)$ converge to the normal distribution

$$F(s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{s} e^{-u^2}\, du.$$

Let $G(z) = z^n$ and let

$$h_n(z) = \int_{-\infty}^{\infty} (z - is)^n\, dF(s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (z - is)^n e^{-s^2}\, ds.$$

We find that this is

$$h_n(z) = (-1)^n 2^{-n} e^{z^2} \frac{d^n}{dz^n}\left(e^{-z^2}\right) = 2^{-n} H_n(z)$$

where $H_n(z)$ is the nth Hermite polynomial (see Szegö [21]). It is a well known fact that the Hermite polynomials have only real zeros all of which are simple. This example gives an alternate proof that the zeros of the Hermite polynomials are real. For the traditional argument involving inner products see [11, Section 10.3] or [21, Section 3.3]. From the point of view of Theorem 1 along with the heuristic of the first example the convolution of z^n with the normal distribution leaves the zeros on the real axis and causes the zeros to repel from each other.

4. If $G \in \mathcal{LP}^*$, then convolution of G with a symmetric distribution not of the type in Theorem 1 or Lemma 2 does not necessarily produce a function with only real zeros. For example, if $G(z) = z^4$, then

$$H(z) = \frac{1}{19}(G(z+i) + 17G(z) + G(z-i))$$

has no real zeros.

Alternately, if G does not satisfy the requisite genus condition then Theorems 1 and 2 do not necessary hold. For example, let $G(z) = ze^{z^2}$. This function is not in $\mathcal{LP}^*$ because it has genus 2 instead of 0 or 1. Then

$$H(z) = \frac{1}{2}(G(z+i) + G(z-i))$$

does not have only real zeros. For example, there is a zero approximately at $z = 0.957505i$.

5. In Theorem 1 let X_i for $i = 1, 2, 3, \ldots$ be the independent random variables taking values $\pm 2^{-i}$ with equal probability. Then the distribution function of the random variable $Y = \sum_{n=1}^{\infty} X_n$ is

$$F(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{1}{2} + \frac{x}{2} & \text{if } -1 \le x < 1 \\ 1 & \text{if } 1 \le x \end{cases}$$

If $G \in \mathcal{LP}^*$ and $G * dF$ is not identically zero then

$$H(z) = (G * dF)(z) = \tfrac{1}{2} \int_{-1}^{1} G(z - is)\, ds$$

also has only real zeros.

If we alter the random variables X_i in this example to take values $\pm 3^{-i}$ then the measure dF will have support on a Cantor middle thirds set instead of an interval!

6. If $G(z)$ has zeros in a strip $-c \le \Im(z) \le c$, under what conditions can we say that the zeros of $(G * dF)(z)$ are in the strip $-a < \Im(z) < a$ for $a \le c$? (A similar question has been considered by Bump et al [2]. See also de Bruijn [3] Theorems 3 and 5.)

7. Can the function

$$\Xi(t) = \xi(1/2 + it) \quad \text{where} \quad \xi(s) = \tfrac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

be realized as a convolution $\Xi(t) = (G*dF)(t)$ where $G(t) \in \mathcal{LP}^*$? This would prove the Riemann Hypothesis! However, it seems unlikely that this approach would be fruitful because the formula (see [22, p. 255])

$$\Xi(t) = 2 \int_0^{\infty} \Phi(u) \cos(ut)\, du$$

with

$$\Phi(u) = 2 \sum_{n=1}^{\infty} (2n^4\pi^2 e^{9u/2} - 3n^2\pi e^{5u/2}) e^{-n^2\pi e^{2u}}$$

suggests that it may be more natural to consider $\Xi(t)$ as a Fourier integral than as a convolution $G * dF$. (For basic properties of $\zeta(s)$ see the book by Titchmarsh [22].)

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