

A EUCLIDEAN RING CONTAINING $\mathbb{Z}[\sqrt{14}]$

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ABSTRACT. We show that the ring $\mathbb{Z}[\sqrt{14}, 1/2]$ is Euclidean and that this ring admits a totally multiplicative Euclidean function. This example is interesting because it is related to the ring $\mathbb{Z}[\sqrt{14}]$ which is conjectured to be Euclidean but for which a proof is lacking.

Considerable effort has been spent in trying to classify the Euclidean number fields (see Lemmermeyer [8]). An integral domain R is called *Euclidean* if there is a function $\phi : R \rightarrow \{0, 1, 2, \dots\}$ such that $\phi(a) = 0$ if and only if $a = 0$ and for every $a, b \in R$, $b \neq 0$, there exist $q, r \in R$ such that $a = bq + r$ with $\phi(r) < \phi(b)$. A number field is said to be Euclidean when its ring of integers is Euclidean.

Of the finite number of known examples of Euclidean number fields many are norm-Euclidean; that is, ϕ is the absolute value of the norm. It is well known that there are only finitely many norm-Euclidean real quadratic number fields. However, Weinberger [11] proved that if one assumes an appropriate Riemann Hypothesis then a ring of algebraic integers which is a principal ideal domain and has infinitely many units is Euclidean. Lenstra [9] extended Weinberger's result to rings of S -integers and gave an explicit Euclidean function. Clark [3] found the first example of a real quadratic number field that is Euclidean but not for the norm, the field being $\mathbb{Q}(\sqrt{69})$. Weinberger's result predicts that $\mathbb{Q}(\sqrt{14})$ as well as many other number fields should also be Euclidean.

Without using a Riemann hypothesis Samuel [10] and Bedocchi [2] provide evidence that suggests $\mathbb{Z}[\sqrt{14}]$ might be Euclidean. In this paper we will show that the related ring $\mathbb{Z}[\sqrt{14}, 1/2]$ is Euclidean with a completely multiplicative Euclidean function. Harper [5], by a more complicated method than that presented here, has shown that $\mathbb{Z}[\sqrt{14}, 1/p]$ is Euclidean for every prime p . However, the Euclidean functions constructed by that argument are not completely multiplicative.

Proposition 1. *Let $\eta = 4 + \sqrt{14}$. (This is the prime above 2 in the ring $\mathbb{Z}[\sqrt{14}]$.) The ring $\mathbb{Z}[\sqrt{14}, 1/2]$ is Euclidean with a completely multiplicative Euclidean function ϕ given by*

$$\phi(x) = |\text{Norm}(x')|$$

where $\text{Norm}(a + b\sqrt{14}) = a^2 - 14b^2$ denotes the norm of an element in $\mathbb{Q}(\sqrt{14})$ and x is written as $x = x'\eta^\alpha$ such that x' contains no power of η .

The proposition says that if 2 is made into a unit then the ring $\mathbb{Z}[\sqrt{14}]$ becomes the Euclidean ring $\mathbb{Z}[\sqrt{14}, 1/2]$. The Euclidean function given above is the absolute norm in the ring $R = \mathbb{Z}[\sqrt{14}, 1/2]$. Thus, $\phi(x) = |R/(x)|$ which is the number of

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number of cosets modulo the principal ideal (x) . This is also called the S -norm on $\mathbb{Q}(\sqrt{14})$ where S is the set consisting of the archimedean primes and the prime $\eta = 4 + \sqrt{14}$. The proof of proposition 1 will follow from lemmas 2 and 3 below.

Each element $\alpha + \beta\sqrt{14} \in \mathbb{Q}(\sqrt{14})$ may be thought of as a point (α, β) in the plane $\mathbb{R}^2$. We will use a geometrical argument for the quadratic form $F(x, y) = x^2 - 14y^2$ which represents the norm of $x + y\sqrt{14}$. Let Λ be the lattice of points with integer coordinates in $\mathbb{R}^2$. Every point in $\mathbb{R}^2/\Lambda$ may be represented by a point in the fundamental domain

$$D_\Lambda = \{(x, y) \mid 0 \leq x < 1 \text{ and } 0 \leq y < 1\}.$$

Let T be the linear transformation T on $\mathbb{R}^2$ given by

$$(x', y') = T(x, y) = (15x + 56y, 4x + 15y).$$

Then $F(x', y') = F(x, y)$. The transformation T is referred to as an *automorph* in [1]. The powers of T are related to the solutions of the Pellian equation $x^2 - 14y^2 = 1$. For a given lattice L in $\mathbb{R}^2$ define the inhomogeneous minimum of a point $(x, y) \in \mathbb{R}^2$ to be

$$M_L(x, y) = \liminf_{(a, b) \in L} |F(x + a, y + b)|.$$

Then it is apparent (as explained in [1]) that $M_L(x, y) = M_L(T(x, y))$. If it were the case that $M_\Lambda(x, y) < 1$ for every $(x, y) \in \mathbb{Q}^2$ we could deduce that $\mathbb{Z}[\sqrt{14}]$ was norm-Euclidean. Although this is not the case, the following lemma shows it is almost true.

Lemma 2. $M_\Lambda(x, y) < 1$ unless $(x, y) \equiv (1/2, 1/2) \pmod{\Lambda}$. In the exceptional case, $M_\Lambda(1/2, 1/2) = 5/4$.

Remark. Lemmas 2 and 3 are given as Propositions 3.1 and 3.2 in [2]; however, the proof given here is simpler.

Proof of Lemma 2. It is sufficient to prove the lemma for points in D_Λ . We first show the lemma is true for points not too close to $(1/2, 1/2)$.

Consider the vectors $\bar{e}_1 = (\sqrt{14/15}, 1/\sqrt{15})$ and $\bar{e}_2 = (\sqrt{14/15}, -1/\sqrt{15})$. Then $\bar{e}_1$ and $\bar{e}_2$ are normalized eigenvectors of T such that $T(\bar{e}_1) = u\bar{e}_1$ and $T(\bar{e}_2) = u^{-1}\bar{e}_2$ where $u = 15 + 4\sqrt{14}$. Let $\epsilon = \frac{1}{2u}$ and let P be the diamond shaped region

$$P = \{(x, y) \in \mathbb{R}^2 \mid (x, y) = (1/2, 1/2) + \alpha\bar{e}_1 + \beta\bar{e}_2, |\alpha| < \epsilon, |\beta| < \epsilon\}.$$

We divide the square D_Λ into very small squares and check by computer that each of the small square has a translate modulo Λ lying strictly between the hyperbolas $F(x, y) = \pm 1$ unless the small square is entirely contained in the region P .

To deal with the exceptional points lying inside P we note that $(1/2, 1/2)$ is a fixed point of T modulo Λ . If $(x, y) = (1/2, 1/2) + \alpha\bar{e}_1 + \beta\bar{e}_2 \in P$ and $\alpha \neq 0$, choose the smallest $k > 0$ such that $|\alpha u^k| > \epsilon$ but $|\alpha u^{k-1}| < \epsilon$. Then $T^k(x, y) \in D_\Lambda \pmod{\Lambda}$, but $T^k(x, y) \notin P \pmod{\Lambda}$. This shows that (x, y) has a representative $(\tilde{x}, \tilde{y})$ with $F(\tilde{x}, \tilde{y}) < 1$ so that $M_\Lambda(x, y) < 1$. If $\beta \neq 0$ we argue similarly using negative powers of T .

The value $M_\Lambda(1/2, 1/2) = 5/4$ dates back to Heinholt [7]. □

The following modification of the previous lemma applies to arithmetic in $\mathbb{Z}[\sqrt{14}]$ modulo an ideal.

Lemma 3. Let $a, b, c, d \in \mathbb{Z}$.

- (1) Suppose c and d are not both even. Then there is a representative $\tilde{a} + \tilde{b}\sqrt{14}$ of $a + b\sqrt{14}$ modulo $c + d\sqrt{14}$ such that

$$N(\tilde{a} + \tilde{b}\sqrt{14}) < N(c + d\sqrt{14}).$$

- (2) If c and d are both even, the first part of the lemma holds except that the smallest representative of $(\frac{c}{2} + \frac{d}{2}\sqrt{14})(1 + \sqrt{14})$ modulo $c + d\sqrt{14}$ has absolute norm $\frac{5}{4}N(c + d\sqrt{14})$.

Proof. Let Λ be the lattice of points with generators $(1, 0)$ and $(0, 1)$. Any point of $\mathbb{R}^2$ modulo Λ may be represented in a fundamental domain

$$D_\Lambda = \{(s, t) \mid 0 \leq s < 1 \text{ and } 0 \leq t < 1\}.$$

Similarly, let Γ be the lattice with generators (c, d) and $(14d, c)$ and fundamental domain

$$D_\Gamma = \{s(c, d) + t(14d, c) \mid 0 \leq s < 1 \text{ and } 0 \leq t < 1\}.$$

Then Λ represents the whole ring $\mathbb{Z}[\sqrt{14}]$, but Γ represents the ideal generated by $c + d\sqrt{14}$. To adapt lemma 2 which applies to the case of arithmetic modulo Λ to the case of arithmetic modulo Γ we map $(x, y) \in D_\Lambda$ to $(x', y') \in D_\Gamma$ by the linear transformation

$$(x', y') = (cx + 14dy, dx + cy).$$

Then the quadratic form $F(x, y) = x^2 - 14y^2$ at (x', y') becomes

$$F(x', y') = (c^2 - 14d^2)(x^2 - 14y^2) = (c^2 - 14d^2)F(x, y).$$

Thus, the linear transformation has simply rescaled $F(x, y)$. The ‘exceptional’ point $(1/2, 1/2) \in D_\Lambda$ maps to the new ‘exceptional’ point $(c/2 + 7d, d/2 + c/2) \in D_\Gamma$. Hence, if $(1/2, 1/2) \neq (x, y) \in D_\Lambda$, then

$$M_\Gamma(x', y') = |c^2 - 14d^2| M_\Lambda(x, y) < |c^2 - 14d^2|$$

whereas

$$M_\Gamma(c/2 + 7d, d/2 + c/2) = |c^2 - 14d^2| M_\Lambda(1/2, 1/2) = \frac{5}{4}|c^2 - 14d^2|.$$

This proves the lemma. $\square$

Proposition 1 is an immediate consequence of lemma 3. If $1/2$ is adjoined to $\mathbb{Z}[\sqrt{14}]$ then the elements 2 and $\eta = 4 + \sqrt{14}$ become units in $\mathbb{Z}[\sqrt{14}, 1/2]$. The ordinary prime 2 ramifies in $\mathbb{Z}[\sqrt{14}]$ and the prime above it is η . Define a completely multiplicative function ϕ on $\mathbb{Z}[\sqrt{14}, 1/2]$ by

$$\phi(x) = |\text{Norm}(x')|$$

where $\text{Norm}(a + b\sqrt{14}) = a^2 - 14b^2$ denotes the norm of an element in $\mathbb{Q}(\sqrt{14})$ and x is written as $x = x'\eta^\alpha$ such that x' contains no power of η . By inverting 2 we eliminate the difficulty caused by the evenness of c and d in the second part of lemma 3. Thus given any $a, b \in \mathbb{Z}[\sqrt{14}, 1/2]$ with $b \neq 0$ there exist q and r such that $a = bq + r$ and $\phi(r) < \phi(b)$. This completes the proof of proposition 1.

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